Classical Dynamics

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Outline

Introduction

- Classical Dynamics: study of motion of interacting particles and bodies
- Main principles
  - Newton's laws (discovered 1665, published 1687)
  - Drawbacks
    - Cumbersome to apply, especially for constrained multi-body systems
    - Difficult to draw conclusion of a general nature

Review of Newtonian Dynamics

- Newton's law for a particle

$$\mathbf{F} = m\mathbf{a}$$

  - $\mathbf{a} =$ acceleration with respect to an inertial observer

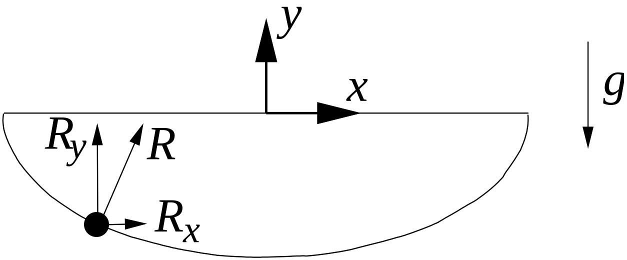
- Newton's law for a system of particles

$$m\ddot{\mathbf{r}}_i = \underbrace{\mathbf{F}_i}_{\text{external force}} + \underbrace{\mathbf{R}_i}_{\text{constraint force}}$$

  - To be solved for $\mathbf{r}_i$ as well as $\mathbf{R}_i$

Example of a Constrained System

- Particle sliding along an elliptical wire under gravity



$$
\begin{aligned}
m\ddot{x} &= R_x \\
m\ddot{y} &= R_y - mg \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{aligned}
$$

- Need to eliminate $R_x,\ R_y,\ y$

Example of a Constrained System (cont'd)

- Normal reaction along inward normal

$$\frac{R_y}{R_x} = \frac{a^2}{b^2}\frac{y}{x} \Longrightarrow$$

$$m\ddot{y} = \frac{a^2}{b^2}\frac{y}{x}m\ddot{x} - mg$$

- Eliminate $y, \ \ddot{y}$

$$y = -\frac{b}{a}\sqrt{a^2 - x^2}$$

$$\ddot{y} = \frac{b\dot{x}^2}{a\sqrt{a^2 - x^2}} + \frac{bx\ddot{x}}{a\sqrt{a^2 - x^2}} + \frac{bx^2\dot{x}^2}{a(a^2 - x^2)^{3/2}}$$

- Final equation

$$\ddot{x}[(b^2 - a^2)x^2 + a^4] + \frac{a^2bx\dot{x}^2}{(a^2 - x^2)} + agx\sqrt{a^2 - x^2} = 0$$

- Point: Newton's law cumbersome to apply to constrained systems

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Lagrangian Dynamics

- From "Treatise on Dynamics", 1687, by Lagrange
- Lagrange's equation of motion in terms of scalar functions like kinetic energy and potential energy
- No constraint forces to account for (conditions apply!)
- Provides an "extension" of Newton's laws

  Every particle constrained to lie on a frictionless surface moves along a geodesic unless acted upon by an external unbalanced force

  - Geodesic: Locally length minimizing curve on a surface

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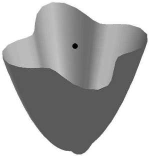
Hamiltonian Dynamics

- Hamilton's principle (1834)
  - *Among all possible motions between two end points, the physical motion renders stationary a certain action integral*

$$\int_{\text{begin}}^{\text{end}} L \, dt$$

  - Nature chooses the "best" path
- Hamilton's equations
  - Reformulation of Lagrange equations
  - Can be used to deduce recurrence without solving the equations



Constraints

- A system of n- particles described by $3n$ coordinates
- System may be constrained by

$$\phi(x) = 0,\ x = [x_1\ x_2\ \cdots\ x_{3n}]^{\mathrm{T}} \in \mathbb{R}^{3n},\ \phi : \mathbb{R}^{3n} \to \mathbb{R}^p$$

- Example: Two particles in a plane connected by a rigid rod

$$\begin{aligned}
\phi_1(x) &\stackrel{\text{def}}{=} (x_1 - x_2)^2 + (x_3 - x_4)^2 - l^2 \\
\phi_2(x) &\stackrel{\text{def}}{=} x_5 \\
\phi_3(x) &\stackrel{\text{def}}{=} x_6
\end{aligned}$$

Holonomic Constraints

- Constraints expressed directly in terms of position
- Described by

$$\phi(t, x) = 0$$

  - Stationary or *scleronomic*: ϕ is independent of time in a suitable inertial frame
  - Moving or *rheonomic*: ϕ depends on time

- Examples:
  - Particles in a plane connected by a rigid rod - scleronomic
  - Particles connected by a rod with specified length variation - rheonomic
  - Spherical pendulum -scleronomic
  - Particle on a rotating hoop - rheonomic

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Configuration Space

- A configuration of a system is a particular arrangement of its various particles that is consistent with the holonomic constraints acting on it
- Configuration space $\mathcal{Q}$ = set of all configurations

$$\mathcal{Q} = \underbrace{\{x \in \mathbb{R}^{3n} : \ \phi(x) = 0\}}_{\text{intersection of hypersurfaces}}$$

- $\mathcal{Q}$ can often be identified with familiar low-dimensional spaces

1 particle in 3D space - $\mathcal{Q} = \mathbb{R}^3$

2 particles in 3D space - $\mathcal{Q} = \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$

1 particle in plane - $\mathcal{Q} = \mathbb{R}^2$

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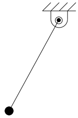
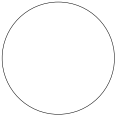
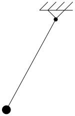
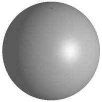
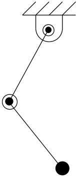
Examples of Configuration Spaces

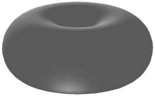

Simple pendulum

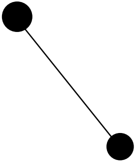

Circle $\mathcal{Q} = S^1$

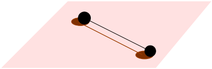

Spherical pendulum


Sphere $\mathcal{Q} = S^2$


Double pendulum


Torus $\mathcal{Q} = S^1 \times S^1$


Dumbbell
$\mathcal{Q} = S^2 \times \mathbb{R}^3$


Planar dumbbell
$\mathcal{Q} = S^1 \times \mathbb{R}^2$

Number of d.o.f $= 3n -$ number of constraints $=$ dimension of $\mathcal{Q}$

Generalized Coordinates

- Need to represent configuration by numbers
  - Example: Cartesian coordinates of all particles in the system
    - ⋆ Not independent in presence of constraints
    - ⋆ May be possible to use fewer quantities
- Generalized coordinates: Any set of quantities that give an unambiguous representation of the configuration of the system
- Independent generalized coordinates
  - Constraints automatically satisfied when expressed in independent generalized coordinates

    Number of independent generalized coordinates = number of d.o.f
  - Can be thought of as curvilinear coordinates on $\mathcal{Q}$

$$q = [q_1 \ \cdots \ q_r]^{\mathrm{T}} \in \mathbb{R}^r$$

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Examples of Independent Generalized Coordinates

- Particle in a plane, $\mathcal{Q} = \mathbb{R}^2,\ q = (q_1, q_2)$ coordinates with respect to any set of independent axes
- Simple pendulum, $\mathcal{Q} = S^1,\ q = \theta$ angle from suitable reference
- Dumbbell in a plane, $\mathcal{Q} = S^1 \times \mathbb{R}^2,\ q = (x, y, \theta)$
- Spherical pendulum, $\mathcal{Q} = S^2,\ q = (\text{latitude}, \text{longitude})$
- Double pendulum, $\mathcal{Q} = S^1 \times S^1,\ q = (\theta_1, \theta_2)$
- Two d.o.f. spring mass system
- Rigid triangle of particles, d.o.f$= 6$

Positions and Generalized Coordinates

- Position of every particle in the system is a function of the generalized coordinates
- Examples:
  - Particle in a plane, $(x, y) = (q_1, q_2)$
  - Simple pendulum, $(x, y) = (\cos q, \sin q)$
  - Dumbbell in a plane

$$(x_1, y_1) = (q_1 - l\cos q_3, q_2 - l\sin q_3)$$

$$(x_2, y_2) = (q_1 + l\cos q_3, q_2 + l\sin q_3)$$

  - Spherical pendulum, $(x, y, z) = (r\cos q_1 \cos q_2, r\cos q_1 \sin q_2, r\sin q_1)$
  - Double pendulum

$$(x_1, y_1) = (l_1 \cos q_1, l_1 \sin q_1)$$

$$(x_2, y_2) = (l_1 \cos q_1 + l_2 \cos q_2, l_1 \sin q_1 + l_2 \sin q_2)$$

Velocities and Generalized Velocities

- Generalized velocities are the rates of change of generalized coordinates

$$\dot{q} = [\dot{q}_1 \ \cdots \ \dot{q}_r]^{\mathrm{T}}$$

- Velocity of every particle is a function of q and $\dot{q}$
  - Particle in plane, $(\dot{x}, \dot{y}) = (\dot{q}_1, \dot{q}_2)$
  - Simple pendulum, $(\dot{x}, \dot{y}) = (-\dot{q}\sin q, \dot{q}\cos q)$
  - Dumbbell in a plane

$$(\dot{x}_1, \dot{y}_1) = (\dot{q}_1 + l\dot{q}_3 \sin q_3, \dot{q}_2 - l\dot{q}_3 \cos q_3)$$

$$(\dot{x}_2, \dot{y}_2) = (\dot{q}_1 - l\dot{q}_3 \sin q_3, \dot{q}_2 + l\dot{q}_3 \cos q_3)$$

  - Spherical pendulum

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}-r\sin q_1 \cos q_2 & -r\cos q_1 \sin q_2\\ -r\sin q_1 \sin q_2 & r\cos q_1 \cos q_2\\ r\cos q_1 & 0\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

Generalized Velocities

$$x_i = x_i(q_1, \ldots, q_r)$$

$$\dot{x}_i = \sum_{j=1}^{r} \frac{\partial x_i}{\partial q_j}(q)\dot{q}_j$$

$$= \left[\frac{\partial x_i}{\partial q}(q)\right]^{\mathrm{T}} \dot{q}$$

$$\underbrace{\frac{\partial x_i}{\partial q}}_{\text{gradient}} : \mathbb{R}^r \to \mathbb{R}^r$$

$$x = x(q)$$

$$\dot{x} = \sum_{j=1}^{r} \frac{\partial x}{\partial q_j}(q)\dot{q}_j$$

$$= \underbrace{\frac{\partial x}{\partial q}(q)}_{\text{Jacobian}} \dot{q}$$

$$\frac{\partial x}{\partial q_j} : \mathbb{R}^r \to \mathbb{R}^{3n}$$

$$\frac{\partial x}{\partial q} : \mathbb{R}^r \to \mathbb{R}^{3n \times r}$$

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Velocities as Tangents to Configuration Space

- Suppose $x(t)$ is a motion that satisfies the constraints

$$\phi_i\left(x(t)\right) = 0$$

- Motion traces a curve on $\mathcal{Q}$, with velocity vector $\dot{x}(t)$

$$0 = \left.\frac{d}{dt}\right|_{t=0} \phi_i(x(t)) = \left[\frac{\partial \phi_i}{\partial x}(x(0))\right]^{\mathrm{T}} \dot{x}(0)$$

$$\frac{\partial \phi_i}{\partial x}(x(0)) = \text{ Normal to } \mathcal{Q} \text{ at } x(0)$$

$$\implies \dot{x}(0) \text{ is tangent to } \mathcal{Q} \text{ at } x(0)$$

  - Configurations are points in $\mathcal{Q}$
  - Motions are curves in $\mathcal{Q}$
  - Rates of change of configurations are tangent vectors to $\mathcal{Q}$

A Basis for Tangent Vector

- Suppose $q_1, \ldots, q_r$ are independent generalized coordinates for system satisfying

$$\phi_i(x) = 0 \Longrightarrow \phi_i(x(q)) = 0 \text{ for all } q \Longrightarrow \left[\frac{\partial \phi_i}{\partial x}(x(q))\right]^{\mathrm{T}} \frac{\partial x}{\partial q_j}(q) = 0$$

$$\frac{\partial \phi_i}{\partial x}(x(q)) = \text{ Normal to } \mathcal{Q} \text{ at } x(q)$$

$$\Longrightarrow \frac{\partial x}{\partial q_j}(q) \text{ is tangent to } \mathcal{Q} \text{ at } x(q)$$

$\frac{\partial x}{\partial q_j}$ is tangent to the curve obtained by varying q_j for fixed values of other q's

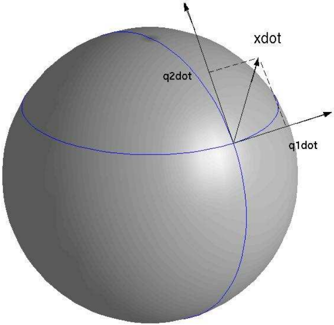
- $\dot{x} \in$ tangent space to $\mathcal{Q}$
- $\frac{\partial x}{\partial q_j}$, $j = 1, \ldots, r$, basis vectors for the tangent space $\mathcal{Q}$
- $\dot{q}_1, \ldots, \dot{q}_r$ components of $\dot{x}$ in this basis

Example

$$\mathcal{Q} = S^2$$

$$x(q) = [\cos q_1 \cos q_2 \;\; \cos q_1 \sin q_2 \;\; \sin q_1]^{\mathrm{T}}$$

$$\frac{\partial x}{\partial q_1} = [-\sin q_1 \cos q_2 \; -\sin q_1 \sin q_2 \;\; \cos q_1]^{\mathrm{T}}, \; \frac{\partial x}{\partial q_2} = [-\cos q_1 \sin q_2 \;\; \cos q_1 \cos q_2 \; 0]^{\mathrm{T}}$$



Non-independent Generalized Coordinates

- General non-holonomic constraint

$$\phi(t, q) = 0$$

- Transformation to Cartesian coordinates

$$x = x(t, q)$$

- System is
  - *Scleronomic* if neither the constraint nor the transformation equations involve time
  - *Rheonomic* otherwise

Differential of a Function

- Given $\psi : \mathcal{Q} \to \mathbb{R}$, $q_0 \in \mathcal{Q}$ and v tangent to $\mathcal{Q}$ at q_0
- Define rate of change of ψ along v at q_0

$$d\psi_{q_0}(v) \stackrel{\text{def}}{=} \left.\frac{d}{dt}\right|_{t=0} \psi(r(t))$$

  - $r(\cdot)$ is any motion starting at q_0 with initial velocity v

$$d\psi_{q_0}(v) = \left[\frac{\partial\psi}{\partial q}(q_0)\right]^{\mathrm{T}} v$$

$$d\psi_q(v) = \frac{\partial\psi}{\partial q_1}(q)dq_1(v) + \cdots + \frac{\partial\psi}{\partial q_r}(q)dq_r(v)$$

- Abbreviated as $d\psi = \dfrac{\partial\psi}{\partial q_1}dq_1 + \cdots + \dfrac{\partial\psi}{\partial q_r}dq_r$
- $d\psi_{(\cdot)}(\cdot)-$ differential of ψ
  - Linear in v at every $q \in \mathcal{Q}$

Velocity Constraints

- Constraints on positions also give rise to constraints on velocities
  - If $\phi = 0$ along a motion, then rate of change of $\phi = 0$ as well
- If admissible motions satisfy $\phi(q) = 0$, then every admissible velocity at $q \in \mathcal{Q}$ satisfies $d\phi_q(v) = 0$
  - Short hand: Configurations satisfy $\phi = 0$, then velocities satisfy $d\phi = 0$
- At each $q \in \mathcal{Q}$, the set of admissible velocities is the linear space

$$\{v : \ d\phi_q(v) = 0\} \quad = \quad \underbrace{\left\{v : \ \left[\frac{\partial \phi}{\partial q}(q)\right]^{\mathrm{T}} v = 0\right\}}_{\text{tangent space to } \mathcal{Q} \text{ at } q}$$

Differential Forms and Velocity Constraints

- A *differential form* is a function of q and v which is linear in v for every fixed q
  - Example: $d\psi_q(v)$ for $\psi : \mathcal{Q} \to \mathbb{R}$
- A general differential form is of the form

$$\begin{aligned} a_q(v) &= a(q)^{\mathrm{T}} v \\ &= a_1(q)v_1 + \cdots + a_r(q)v_r \end{aligned}$$

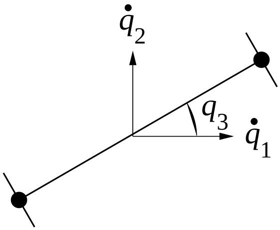
  - Short hand: $a = a_1 dq_1 + \cdots + a_r dq_r$
- Differential form a is *exact* if $a = d\psi$ for some function ψ
- A general linear velocity constraint is of the form

$$a_q(v) = 0, \text{ that is, } a_1 dq_1 + \cdots + a_r dq_r = 0$$

- Does this velocity constraint arise from a position constraint?
  - Yes, if a is exact
  - No in general

Velocity Constraints: An Example

- Dumbbell on a plane with knife edges orthogonal to the dumbbell



  - Knife edges restrict velocity at each particle to be perpendicular to the rod

- Along any motion of the dumbbell

$$\dot{q}_1 \cos q_3 + \dot{q}_2 \sin q_3 = 0$$

- That is, every admissible velocity vector satisfies

$$\underbrace{\cos q_3 \ dq_1 + \sin q_3 \ dq_2}_{\text{differential form with } a(q)=[\cos q_3 \ \ \sin q_3 \ \ 0]^{\mathrm{T}}} = 0$$

Example: A Few Questions

- Configuration space 3D
- Set of allowable velocities at each configuration is a 2D linear space
  - Is there a family of 2D surfaces tangent to all these linear spaces?
  - Does the velocity constraint restrict configurations that can be reached from a given initial configuration?
- Yes, if a is exact, that is, $a = d\psi$ for some ψ

$$a = 0 \quad \Rightarrow d\psi = 0 \quad \Rightarrow \psi = \text{constant}$$

- Check: If $a = [\cos q_3 \;\; \sin q_3 \;\; 0]^{\mathrm{T}} = \left[\frac{\partial \psi}{\partial q_1} \;\; \frac{\partial \psi}{\partial q_2} \;\; \frac{\partial \psi}{\partial q_3}\right]^{\mathrm{T}}$, then

$$\cos q_3 = \frac{\partial}{\partial q_3}\left(\frac{\partial \psi}{\partial q_2}\right) \neq \frac{\partial}{\partial q_2}\left(\frac{\partial \psi}{\partial q_3}\right) = 0 \;!$$

- a is not exact

A Necessary Condition for Exactness

- If $a = a_1 dq_1 + a_2 dq_2 + a_3 dq_3$ is exact, then

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \nabla\psi \text{ for some } \psi$$

$\therefore$ curl $a = 0$, that is,

$$\frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i} = 0,\ i \neq j,\ i, j = 1, 2, 3.$$

- In higher dimensions, if $a = a_1 dq_1 + \cdots + a_r dq_r$ is exact, then

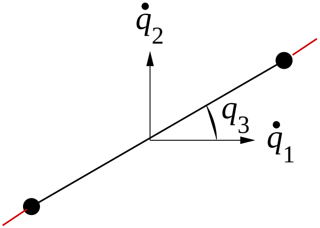
$$\frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i} = 0,\ i \neq j,\ i, j = 1, \ldots, r.$$

  - Sufficient under additional conditions

- Question: If a is not exact, does it follow that the configuration space is not restricted?

Velocity Constraints: Another Example

- Dumbbell on a plane with knife edges parallel to the dumbbell



  - Knife edges restrict velocity at each particle to lie along the rod

- Velocity constraint at the center of the dumbbell

$$\sin q_3 \; dq_1 - \cos q_3 \; dq_2 = 0$$

- Not exact, but dumbbell restricted to move in a straight line

Integrability

- Even if a is not exact, a may be *integrable*, that is, there may exist an integrating factor $\eta: \mathcal{Q} \to \mathbb{R}$ such that $\eta a = \eta a_1 dq_1 + \eta a_2 dq_2 + \eta a_3 dq_3$ is exact
- If a is integrable, there exist functions η and ψ such that $a = \frac{1}{\eta} d\psi$
  - Abuse of notation: Think of a as a vector field $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$\begin{aligned} a &= \tfrac{1}{\eta}\nabla\psi \\ \text{curl } a &= -\tfrac{1}{\eta^2}(\nabla\eta \times \nabla\psi) + \tfrac{1}{\eta}\underbrace{\text{curl}\nabla\psi}_{=0} \\ &= -\tfrac{1}{\eta}(\nabla\eta \times a) \end{aligned}$$

$$\therefore \; a \cdot \text{curl } a = 0$$

$$a_1\left(\frac{\partial a_3}{\partial q_2} - \frac{\partial a_2}{\partial q_3}\right) + a_2\left(\frac{\partial a_1}{\partial q_3} - \frac{\partial a_3}{\partial q_1}\right) + a_3\left(\frac{\partial a_2}{\partial q_1} - \frac{\partial a_1}{\partial q_2}\right) = 0$$

Non-Holonomic Constraints

- If a velocity constraint a is integrable, then ηa is exact for some η
  - a and ηa define the same set of allowable velocities
  - The velocity constraint can be "integrated" to yield a position constraint
- If the velocity constraint is not integrable, then it does not restrict configurations to a lower dimensional subset
- Such a constraint is truly "non-holonomic"
- Issues
  - Necessary and sufficient conditions for integrability
  - Multiple velocity constraints
  - Higher dimensions

} Calculus of differential forms

Dumbbell with Perpendicular Knife-Edges

$$a = [\sin q_3 \ \ \cos q_3 \ \ 0]^{\mathrm{T}}, \ \text{curl } a = a$$

$$a \cdot \text{curl } a = 1 \neq 0$$

- Constraint is not integrable. Does not restrict attainable configurations
  - Can we explicitly work out paths between configurations?

Dumbbell with Perpendicular Knife-Edges

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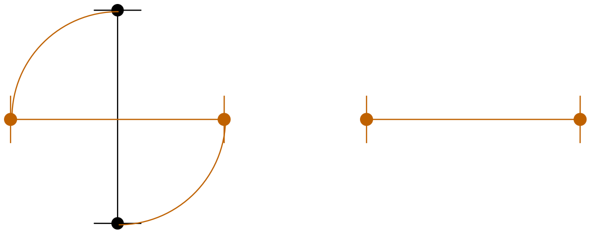

Dumbbell with Perpendicular Knife-Edges

$$a = [\sin q_3 \;\; \cos q_3 \;\; 0]^{\mathrm{T}}, \;\; \text{curl } a = a$$

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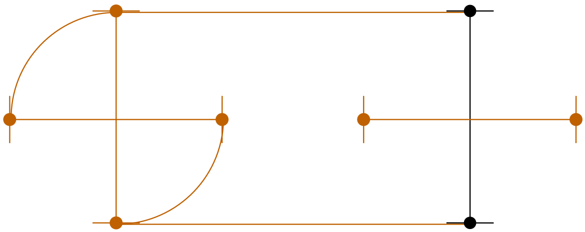

Dumbbell with Perpendicular Knife-Edges

$$a = [\sin q_3 \;\; \cos q_3 \;\; 0]^{\mathrm{T}}, \;\; \text{curl } a = a$$

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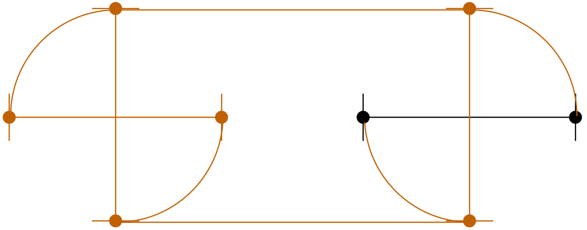

Dumbbell with Perpendicular Knife-Edges

$$a = [\sin q_3 \;\; \cos q_3 \;\; 0]^{\mathrm{T}}, \;\; \text{curl } a = a$$

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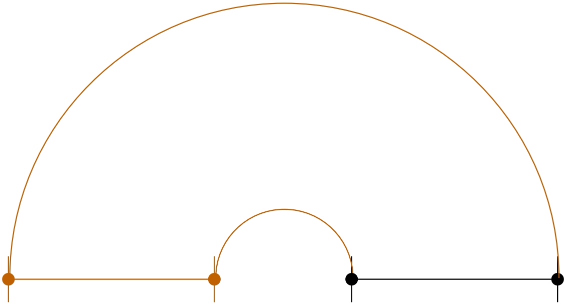
  - Can we explicitly work out paths between configurations?

Dumbbell with Perpendicular Knife-Edges

$$a = [\sin q_3 \;\; \cos q_3 \;\; 0]^{\mathrm{T}}, \; \text{curl } a = a$$

$$a \cdot \text{curl } a = 1 \neq 0$$

- Constraint is not integrable. Does not restrict attainable configurations
  - Can we explicitly work out paths between configurations?



Examples of Non-Holonomic Systems

- Cars, cars with trailers
  - No sideways velocity, but sideways displacement possible
- Snakes, snake board
  - Periodic shape change leads to linear motion
- Ball on a plate
  - Periodic position change leads to a periodic orientation change
- Multi-body space systems
  - Falling cats, divers
  - Periodic shape change leads to orientation change
- Rattle backs, wobble stones, tippy tops

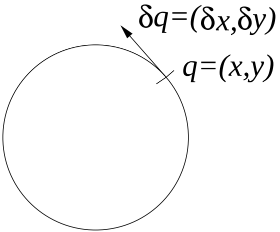
Unilateral Constraints

- Bilateral constraints are equality constraints of the kind $\phi = 0$ or $a = 0$
  - Positions and/or velocities constrained to a lower dimensional surface
- Inequality constraints of the form $\phi \geq 0,\ a \geq 0$ also possible
- Example: Particle moving outside a sphere
  - Position constraint $\phi(x, y, z) = x^2 + y^2 + z^2 - r^2 \geq 0$
  - Velocity constraint $d\phi_q(v) \geq 0$ whenever $\phi(q) = 0$
- Any motion has two kinds of segments
  - Particle moves in contact with the sphere
  - Particle moves out of contact with the sphere
- Each segment can be solved by using initial conditions from the previous segment
  - Monitor constraint force to detect loss of contact
  - Monitor constraint function to detect contact

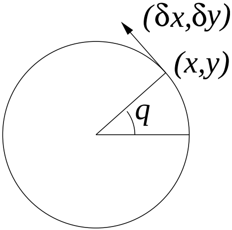
Virtual Displacement

- Consider a scleronomic system described using generalized coordinates $q_1, \ldots, q_r$ subject to
$$\phi(q) = 0,\ a_q(v) = 0$$
- A *virtual displacement* at $q \in \mathcal{Q}$ is a vector $\delta q \in \mathbb{R}^r$ satisfying
$$d\phi_q(\delta q) = 0,\ a_q(\delta q) = 0$$
  - A tangent vector to the configuration space lying in the set of admissible velocities
- Particles of the system undergo virtual displacements along δq
$$x = x(q) \Longrightarrow \quad \delta x = \frac{\partial x}{\partial q}(q)\delta q$$
  - δx is linear approximation to the change in x when q changes to $q + \delta q$
- A virtual displacement is also an admissible velocity

Example



$$\begin{aligned} \phi(q) &= x^2 + y^2 - 1 &= 0 \\ d\phi_q(\delta q) &= x\ \delta x + y\ \delta y &= 0 \\ \delta q &= \alpha[y\ \ -x]^{\mathrm{T}} \end{aligned}$$



$$x = \cos q, \quad y = \sin q$$
$$\delta x = -\sin q\ \delta q, \quad \delta y = \cos q\ \delta q$$

Virtual Displacement for Rheonomic Systems

- Consider a system subject to time varying position and velocity constraint
- Set of virtual displacements changes every instant
- At each instant, the set of virtual displacements is the set of tangent vectors to the instantaneous surface

$$\phi(q, t) = 0$$

that satisfy the instantaneous velocity constraint

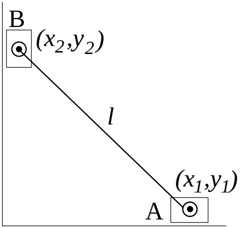
$$a_1(q, t)dq_1 + \cdots + a_r(q, t)dq_r = 0$$

- If $x = x(q, t)$, the virtual displacements of the particles are given by

$$\delta x = \frac{\partial x}{\partial q}(q, t)\ \delta q$$

  - Treat time as frozen to calculate instantaneous virtual displacement
  - Virtual displacements are not actual velocities

Virtual Displacements: Example 1



Constraints

$$y_1 = 0,\ x_2 = 0$$
$$x_1^2 + y_2^2 = l^2$$

- $\mathcal{Q} =$ circle in the x_1–y_2 plane in (x_1, y_1, x_2, y_2) space
- Along any tangent vector $(\delta x_1, \delta y_1, \delta x_2, \delta y_2)$ to this circle

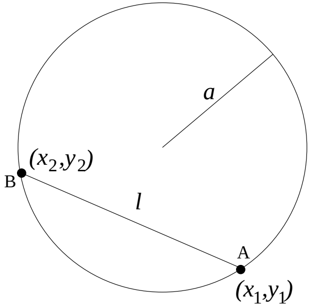
$\delta y_1 = 0$ (virtual displacement of A is horizontal)

$\delta x_2 = 0$ (virtual displacement of B is vertical)

$$x_1\ \delta x_1 + y_2\ \delta y_2 = 0$$

$$\frac{(y_2 - y_1)}{(x_2 - x_1)} \frac{(\delta y_2 - \delta y_1)}{(\delta x_2 - \delta x_1)} = -1$$ (relative virtual displacement orthogonal to rod)

Virtual Displacement: Example 2



$$x_1^2 + y_1^2 - a^2 = 0$$
$$x_2^2 + y_2^2 - a^2 = 0$$
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 - l^2 = 0$$

- Along any virtual displacement

$$\left.\begin{aligned} x_1\ \delta x_1 + y_1\ \delta y_1 &= 0 \\ x_2\ \delta x_2 + y_2\ \delta y_2 &= 0 \end{aligned}\right\}$$ virtual displacements of A,B tangent to the circle

$(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) = 0$ — relative virtual displacement perpendicular to rod

Virtual Work

- Consider a n- particle system having coordinates $x \in \mathbb{R}^{3n}$
- Components of total forces acting on the particles $F \in \mathbb{R}^{3n}$
- Along a virtual displacement $\delta q \in \mathbb{R}^r$ of the system

$$\delta x = \frac{\partial x}{\partial q}(q,t)\ \delta q$$

- *Virtual work* of the system of forces along the virtual displacement δq is defined as

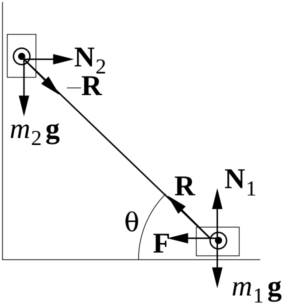
$$\delta W = F^{\mathrm{T}} \delta x = F^{\mathrm{T}} \frac{\partial x}{\partial q}(q,t)\ \delta q$$

  - Note: no actual motion or displacement
  - Linear in δq at each q, t
  - Inner product of $F \in \mathbb{R}^{3n}$ with the vector $\delta x \in \mathbb{R}^{3n}$ tangent to $\mathcal{Q}$

- $\delta \mathbf{r}_i =$ virtual displacement of i^{th} particle, $\mathbf{F}_i =$ net force on i^{th} particle

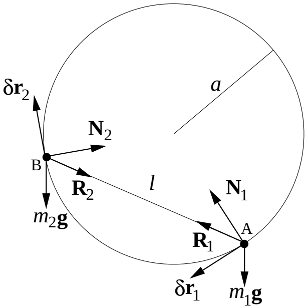
$$\delta W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i$$

Virtual Work: Example 1



$$
\begin{aligned}
\delta x &= [\delta x_1 \quad \delta y_1 \quad \delta x_2 \quad \delta y_2]^{\mathrm{T}} \\
F &= [-F - R\cos\theta \quad N_1 - m_1 g \quad N_2 + R\cos\theta \quad - m_2 g - R\sin\theta]^{\mathrm{T}} \\
\delta W &= -F\,\delta x_1 - m_2 g\,\delta y_2
\end{aligned}
$$

Virtual Work: Example 2



$$\begin{aligned} \mathbf{N}_2 \cdot \delta\mathbf{r}_2 &= \mathbf{N}_1 \cdot \delta\mathbf{r}_1 &= 0 \\ \mathbf{R}_2 \cdot \delta\mathbf{r}_2 + \mathbf{R}_1 \cdot \delta\mathbf{r}_1 &= \mathbf{R}_2 \cdot (\delta\mathbf{r}_2 - \delta\mathbf{r}_1) &= 0 \end{aligned}$$

$$\delta W = -m_1 g\ \delta y_1 - m_2 g\ \delta y_2$$

Workless Constraints

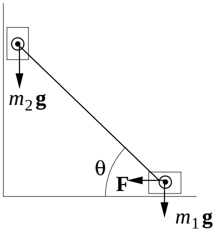
- A bilateral constraint is *workless* if the virtual work of the corresponding constraint forces is zero for every virtual displacement of the system
- Main examples
  - Rigid interconnections between particles
    - Constraint forces equal and opposite along the interconnection
    - Relative virtual displacement orthogonal to the interconnection
  - Sliding motion on a frictionless surface
    - Constraint force normal to the surface
    - Virtual displacement at point of contact tangent to surface
  - Rolling without slipping
    - Virtual displacement of point of contact is zero

Equilibrium Configurations

- A configuration in which the total force $(\mathbf{F} + \mathbf{R})$ acting on each particle is zero
- A system in an equilibrium configuration at rest remains in that configuration
- Principle of virtual work: which configurations are equilibrium configurations?

  A configuration q of a scleronomic system having workless constraints is an equilibrium configuration if and only if the virtual work of external (nonconstraint) forces along every virtual displacement at q is zero
- Example: Spherical pendulum

Principle of Virtual Work: Example

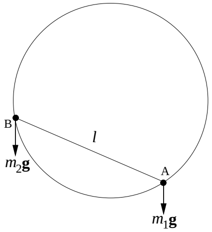


$$\delta W = -F\ \delta x_1 - m_2 g\ \delta y_2 = 0$$

for every $\delta x_1, \delta y_2$ satisfying

$$\cos\theta\ \delta x_1 + \sin\theta\ \delta y_2 = 0$$

$$\implies \tan\theta = \frac{m_2 g}{F}$$



$$\delta W = -m_1 g\ \delta y_1 - m_2 g\ \delta y_2 = 0$$

for every $\delta x_1, \delta y_1, \delta x_2, \delta y_2$ satisfying

$$x_1\ \delta x_1 + y_1\ \delta y_1 = 0$$

$$x_2\ \delta x_2 + y_2\ \delta y_2 = 0$$

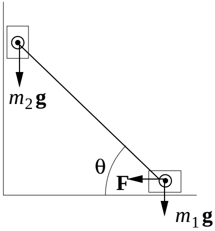
$$(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) = 0$$

$$\implies m_1 x_1 + m_2 x_2 = 0$$

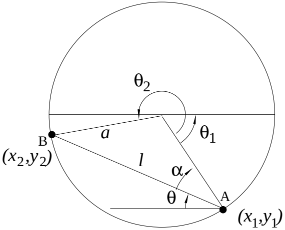
Generalized Forces

- Principle of virtual work in Cartesian coordinates $\delta W = F^{\mathrm{T}}\delta x = 0$ for every $\delta x \in \mathbb{R}^n$ satisfying $\dfrac{\partial \phi(x)}{\partial x}(x)\delta x = 0$
- Problem: Components of δx are not independent. Tedious to apply
- Solution: Write principle of virtual work using generalized coordinates
$$\delta W = F^{\mathrm{T}}\delta x = \left(F^{\mathrm{T}}\frac{\partial x(q)}{\partial q}(q)\right)\delta q$$
- Define *generalized force* $Q \stackrel{\text{def}}{=} \left[\dfrac{\partial x(q)}{\partial q}(q)\right]^{\mathrm{T}} F$
- Generalized force along q_j, $Q_j = \displaystyle\sum_{i=1}^{3n} F_i \frac{\partial x_i}{\partial q_j}$
$$\delta W = Q^{\mathrm{T}}\delta q$$
- If $q_1, \ldots, q_r$ are independent generalized coordinates, then δq are unconstrained
- Principle of virtual work: A system is in equilibrium if and only if the generalized applied forces along a set of independent generalized coordinates are zero.
    - Position constraints only

Example



$$Q_\theta = l(F\sin\theta - m_2 g\cos\theta) = 0$$
for equilibrium



$$\begin{aligned} x_1 &= a\cos\theta_1 &= a\cos(\theta+\alpha) \\ y_1 &= -a\sin\theta_1 &= -a\sin(\theta+\alpha) \\ x_2 &= -a\cos\theta_2 &= -a\cos(\theta-\alpha) \\ y_2 &= -a\sin\theta_2 &= a\sin(\theta-\alpha) \end{aligned}$$

$$Q_\theta = -g(m_A x_1 + m_B x_2)$$

Conservative Forces

- Consider a particle that moves under the influence of a position dependent force $\mathbf{F}$

$$F_x = -\frac{\partial V}{\partial x}(x, y, z), F_y = -\frac{\partial V}{\partial y}(x, y, z), F_z = -\frac{\partial V}{\partial z}(x, y, z)$$

  where V is a function of position only

- Work done along a path $\mathbf{r}(t)$

$$\begin{aligned} &= \int_0^t \mathbf{F}(\mathbf{r}(\tau)) \cdot \dot{\mathbf{r}}(\tau) d\tau &=& -\int_0^t \left(\frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} \right) d\tau \\ &= -\int_0^t \frac{d}{d\tau}[V(\mathbf{r}(\tau))] d\tau &=& -V(\mathbf{r}(t)) + V(\mathbf{r}(0)) \end{aligned}$$

- Work done depends on endpoints, not on the path or the time taken
  - Work done along closed curve $= 0$
- Such forces are *conservative* forces
- Note: Force is not conservative if potential is time dependent

Principle of Virtual Work for Conservative Systems

- Consider a system of n- particles with applied forces given by
$$F_i = -\frac{\partial V}{\partial x_i}(x_1, \ldots, x_{3n})$$
- Work done along a path $x(t)$
$$= \int_0^t F(x(\tau))^{\mathrm{T}} \dot{x}(\tau) d\tau = -\int_0^t \left(\frac{\partial V}{\partial x}(x(\tau))\right)^{\mathrm{T}} \dot{x}(\tau) d\tau = V(x(0)) - V(x(t))$$
- Can consider V as a function of q, since $V = V(x),\ x = x(q)$
$$V(q) \stackrel{\text{def}}{=} V(x(q))$$
- Generalized forces
$$\begin{aligned} Q &= \left(\frac{\partial x}{\partial q}(q)\right)^{\mathrm{T}} F &= -\frac{\partial x}{\partial q}(q)^{\mathrm{T}} \frac{\partial V}{\partial x} \\ Q_j &= -\sum_{i=1}^{3n} \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_j} &= -\frac{\partial V}{\partial q_j} \end{aligned}$$
- Principle of virtual work: A holonomic, scleronomic, conservative system remains in equilibrium only at a stationary point of the potential function

D'Alembert's Principle

- Consider a system of n- particles. The motion satisfies $m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{R}_i$ at every instant
- At every t, along every virtual displacement of the system, we have

$$\sum_{i=1}^{n}(\underbrace{\mathbf{F}_i}_{\text{applied}} + \underbrace{\mathbf{R}_i}_{\text{constraint}} - \underbrace{m_i\ddot{\mathbf{r}}_i}_{\text{inertial}}) \cdot \delta\mathbf{r}_i = 0$$

- For workless constraints,

$$\delta W = \sum_{i=1}^{n}(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i) \cdot \delta\mathbf{r}_i = 0$$

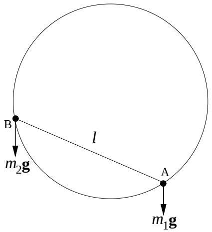
- D'Alembert's principle: *The accelerations along a motion are such that the virtual work done by applied and inertial forces along any virtual displacement is zero*
  - Note: Applies to all workless constraints, scleronomic or rheonomic, unlike principle of virtual work

Jean le Rond d'Alembert

1717-1783

- d'Alembert's solution to wave equation
- d'Alembert's ratio test
- d'Alembert's paradox

d'Alembert's Principle: A Scleronomic Example



$$\delta W = -(m_1 g + m_1 \ddot{y}_1)\delta y_1 - m_1 \ddot{x}_1 \delta x_1 - (m_1 g + m_1 \ddot{y}_2)\delta y_1 - m_1 \ddot{x}_2 \delta x_2 = 0$$

For every $(\delta x_1, \delta y_1, \delta x_2, \delta y_2)$ satisfying

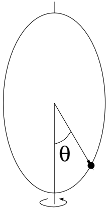
$$x_1 \delta x_1 + y_1 \delta y_1 = 0,\ x_2 \delta x_2 + y_2 \delta y_2 = 0,\ x_2 \delta y_1 - x_1 \delta y_2 = 0$$

- Eliminate $\delta x_1, \delta x_2$

$$m_1(x_1 g + x_1 \ddot{y}_1 - y_1 \ddot{x}_1) + m_2(x_2 g + x_2 \ddot{y}_2 - y_2 \ddot{x}_2) = 0$$

- Constraint forces eliminated, but not the constraint
- Use generalized coordinates

d'Alembert's Principle: A Rheonomic Example



$$x = r\sin\theta\cos\omega t,\ \ y = r\sin\theta\sin\omega t,\ \ z = -r\cos\theta$$

$$\delta W = -(mg + m\ddot{z})\delta z - m\ddot{y}\delta y - m\ddot{x}\delta x = 0$$

where $\delta x = r\cos\theta\cos\omega t\ \delta\theta,\ \ \delta y = r\cos\theta\sin\omega t\ \delta\theta,\ \ \delta z = r\sin\theta\ \delta\theta$

- Substitute for $\ddot{x}, \ddot{y}, \ddot{z}$

$$\ddot{\theta} - \omega^2\sin\theta\cos\theta + \frac{g}{r}\sin\theta = 0$$

- Cumbersome to eliminate the constraint
- Need a general procedure to eliminate constraints and constraint forces by combining generalized coordinates with D'Alembert's principle

Eliminate Constraints

- Eliminate constraints from

$$\sum_{i=1}^{3n}(F_i - m_i\ddot{x}_i)\delta x_i = 0$$

- Suppose $x_i = x_i(q_1, \ldots, q_r, t),\ i = 1, \ldots, 3n$

$$\delta x_i = \sum_{j=1}^{r}\frac{\partial x_i}{\partial q_j}\delta q_j,\ \dot{x}_i = \sum_{j=1}^{r}\frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}$$

$$\sum_{i=1}^{3n}F_i\delta x_i = \sum_{i=1}^{3n}F_i\left(\sum_{j=1}^{r}\frac{\partial x_i}{\partial q_j}\delta q_j\right) = \sum_{j=1}^{r}\left(\sum_{i=1}^{3n}F_i\frac{\partial x_i}{\partial q_j}\right)\delta q_j = \sum_{j=1}^{r}Q_j\delta q_j$$

$$-\sum_{i=1}^{3n}m_i\ddot{x}_i\delta x_i = -\sum_{i=1}^{3n}m_i\ddot{x}_i\left(\sum_{j=1}^{r}\frac{\partial x_i}{\partial q_j}\delta q_j\right) = -\sum_{j=1}^{r}\left(\underbrace{\sum_{i=1}^{3n}m_i\ddot{x}_i\frac{\partial x_i}{\partial q_j}}_{\text{gen. inertia force along } q_j}\right)\delta q_j$$

Elimination of Constraints (cont'd)

- Two identities

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial \dot{x}_i}{\partial q_j}, \quad \frac{\partial x_i}{\partial q_j} = \frac{\partial \dot{x}_i}{\partial \dot{q}_j}$$

$$\begin{aligned}
\ddot{x}_i \frac{\partial x_i}{\partial q_j} &= \frac{d}{dt}\left(\dot{x}_i \frac{\partial x_i}{\partial q_j}\right) - \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) \\
&= \frac{d}{dt}\left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right) - \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j} \\
&= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{x}_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{x}_i^2\right) \\
\therefore \quad \sum_{i=1}^{3n} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\sum_{i=1}^{3n}\frac{1}{2}m_i\dot{x}_i^2\right) - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{3n}\frac{1}{2}m_i\dot{x}_i^2\right)
\end{aligned}$$

Lagrange's Equations

- Total kinetic energy $T(q, \dot{q}, t) = \sum_{i=1}^{3n} \frac{1}{2} m_i \dot{x}_i^2(q, \dot{q}, t)$

$$\sum_{i=1}^{3n} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}$$

- d'Alembert's principle implies

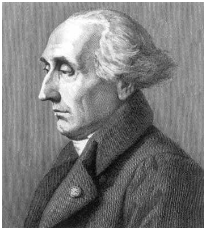
$$\sum_{j=1}^{r} \left[Q_j - \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} \right] \delta q_j = 0$$

- For a holonomic system described by independent generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \; j = 1, \ldots, r$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q}$$

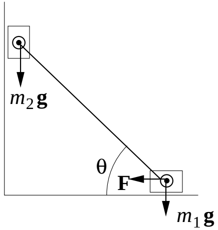
Joseph-Louis Lagrange



Joseph-Louis Lagrange
1736-1813

- Vibrations
- Calculus of variations
- Linear ODE's
- Three-body problem
- Number theory
- Lagrange interpolation
- Mechanics

Lagrange's Equations: An Example



$$x_1 = l\cos\theta$$
$$y_2 = l\sin\theta$$

Lagrange's Equations for Conservative Systems

$$Q = -\frac{\partial V}{\partial q},\, V = V(q)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0$$

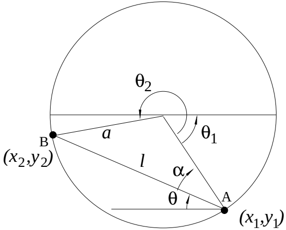
- Define *Lagrangian* $L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

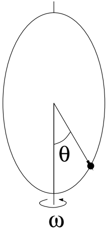
- In general, $Q = -\dfrac{\partial V}{\partial q} + Q_{\text{nc}}$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q_{\text{nc}}}$$

Examples



$$x_1 = a\cos(\theta+\alpha)$$
$$y_1 = -a\sin(\theta+\alpha)$$
$$x_2 = -a\cos(\theta-\alpha)$$
$$y_2 = a\sin(\theta-\alpha)$$



$$x = r\sin\theta\cos\omega t$$
$$y = r sin\theta\sin\omega t$$
$$z = -r\cos\theta$$

Form of the Kinetic Energy

$$x = x(q,t)$$

$$\dot{x} = \frac{\partial x}{\partial q}(q,t)\dot{q} + \frac{\partial x}{\partial t}(q,t)$$

$$T(q,\dot{q},t) = \sum_{i=1}^{3n} \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2}\dot{x}^{\mathrm{T}} J \dot{x}, \;\; J = \mathrm{diag}\{m_1, \ldots, m_{3n}\}$$

$$\begin{aligned} T(q,\dot{q},t) &= \frac{1}{2}\dot{q}^{\mathrm{T}} \underbrace{\left[\frac{\partial x}{\partial q}^{\mathrm{T}} J \frac{\partial x}{\partial q}\right]}_{M(q,t)} \dot{q} + \underbrace{\left[\frac{\partial x}{\partial q}^{\mathrm{T}} J \frac{\partial x}{\partial q}\right]}_{a^{\mathrm{T}}(q,t)} \dot{q} + \frac{1}{2}\frac{\partial x}{\partial t}^{\mathrm{T}} J \frac{\partial x}{\partial t} \\ &= T_2 + T_1 + T_0 \end{aligned}$$

- M— symmetric inertia matrix, positive-definite at every q, t
- For a scleronomic system, $T = T_2$

Form of the Equations

- *Generalized momentum* along q_j is

$$p_j = \frac{\partial T}{\partial \dot{q}_j}, \quad j = 1, \ldots, r$$

$$p = \frac{\partial T}{\partial q} = M(q,t)\dot{q} + a(q,t)$$

- Lagrange's equations

$$\dot{p} - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q$$

  - Linear in $\ddot{q}$
  - Coefficient matrix of $\ddot{q}$ is $M(q,t)$, invertible
  - Can be solved for accelerations to yield

$$\ddot{q} + f(q, \dot{q}, t) = 0$$

d'Alembert's Principle with Velocity Constraints

- An n- particle system subject to m velocity constraints

$$a_i^{\mathrm{T}}\dot{q} + a_{it} = 0,\ i = 1, \ldots, m$$

$$A\dot{q} + b = 0,\ A = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^{\mathrm{T}},\ b = \begin{bmatrix} a_{1t} & \cdots & a_{mt} \end{bmatrix}^{\mathrm{T}}$$

- Virtual displacements satisfy

$$A\delta q = 0$$

- d'Alembert's principle

$$\left(Q - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q}\right)^{\mathrm{T}} \delta q = 0$$

for every δq satisfying $A\delta q = 0$

Lagrange's Equations with Velocity Constraints

$$\text{rank}A = \text{rank}\begin{bmatrix} A \\ Q - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} \end{bmatrix}$$

$$Q - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} \in \text{span of rows of } A$$

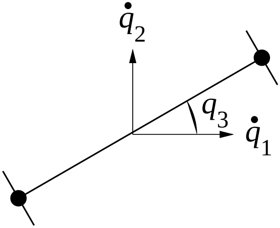
- For every t, there exist scalars $\lambda_1(t), \ldots, \lambda_m(t)$ such that

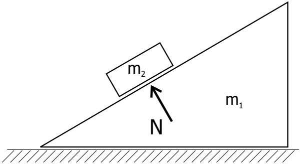
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} - Q = \lambda_1 a_1 + \cdots + \lambda_m a_m \qquad (r \text{ equations})$$

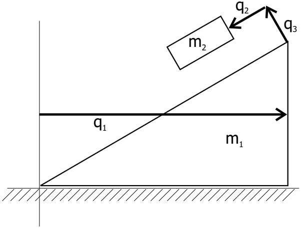
$$A\dot{q} + b = 0 \qquad (m \text{ equations})$$

- $C = \lambda_1 a_1 + \cdots + \lambda_m a_m$ is the constraint force
  - Check: $C^{\mathrm{T}}\delta q = 0$ for every virtual displacement

Examples







Constants of Motion and Integration

- Example: a simple pendulum

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$E(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1 - \cos\theta) = c$$

$$\dot{\theta} = \sqrt{c - \frac{2g}{l}(1 - \cos\theta)}$$

  - θ can be obtained by direct integration (in terms of Jacobi elliptic integrals)
  - E is a *first integral*, an *integral of motion*, a *constant of motion*
- A first integral is a function $f(q, \dot{q}, t)$ such that along any motion, $f(q(t), \dot{q}(t), t)$=constant

$$\frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial \dot{q}}\ddot{q} + \frac{\partial f}{\partial t} = 0$$

- For a 1-d-o-f system, a first integral reduces the problem to an integration (quadrature)
- A n-d-o-f system, having n first integrals can be solved by quadratures
  Examples: two-body problem, free rigid body

Cyclic Coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,\ j = 1, \ldots, r$$

- If $\frac{\partial L}{\partial q_j} = 0$, that is, L is independent of q_j, then

$$p_j = \frac{\partial L}{\partial q_j} = \text{constant}$$

- If $\frac{\partial L}{\partial q_j} = 0,\ q_j$ is an *ignorable* or *cyclic* coordinate
- Fact: The generalized momentum corresponding to an ignorable coordinate is a first integral

Example: Kepler Problem

- Motion under inverse-square attraction to a fixed center

$$T = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2),\ v = -\frac{\mu}{r},\ L = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2) + \frac{\mu}{r}$$
$$\frac{\partial L}{\partial \theta} \equiv 0,\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = m\beta = \text{constant}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Longrightarrow \ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$$

- Substitute for $\dot{\theta}$

$$\ddot{r} - \frac{\beta^2}{r^3} + \frac{\mu}{r^2} = 0$$

  - Solve for r independent of θ, then integrate $\dot{\theta} = \frac{\beta}{r^2}$

- Reduce the order to solve for r, perform a quadrature for θ

- Question: Can we do this as a general procedure?

Routhian Reduction

$$
\begin{aligned}
L &= L(q_{k+1}, \ldots, q_r, \dot{q}_1, \ldots, \dot{q}_r, t) \\
&= L(q_{\mathsf{n}}, \dot{q}_{\mathsf{i}}, \dot{q}_{\mathsf{n}}, t)
\end{aligned}
$$

$$
p = \begin{bmatrix} p_{\mathsf{i}} \\ p_{\mathsf{n}} \end{bmatrix} = \begin{bmatrix} M_1 & M_{12} \\ M_{12}^{\mathrm{T}} & M_2 \end{bmatrix} \begin{bmatrix} \dot{q}_{\mathsf{i}} \\ \dot{q}_{\mathsf{n}} \end{bmatrix} + \begin{bmatrix} a_{\mathsf{i}} \\ a_{\mathsf{n}} \end{bmatrix}
$$

- M positive definite $\Longrightarrow M_1$ positive definite (hence invertible)
- Solve for $\dot{q}_{\mathsf{i}}$ in terms of $p_{\mathsf{i}}, \dot{q}_{\mathsf{n}}, q_{\mathsf{n}}, t$

$$
\dot{q}_{\mathsf{i}} = M_1^{-1} p_{\mathsf{i}} - M_1^{-1} M_{12} \dot{q}_{\mathsf{n}} - M_1^{-1} a_{\mathsf{i}}
$$

- Define *Routhian*

$$
R(q_{\mathsf{n}}, \dot{q}_{\mathsf{n}}, p_{\mathsf{i}}, t) = \underbrace{L(q_{\mathsf{n}}, \dot{q}_{\mathsf{i}}, \dot{q}_{\mathsf{n}}, t) - p_{\mathsf{i}}^{\mathrm{T}} \dot{q}_{\mathsf{i}}}_{\text{substitutefor } \dot{q}_{\mathsf{i}}}
$$

Routhian Reduction (cont'd)

$$\frac{\partial R}{\partial q_j} = \frac{\partial L}{\partial q_j} + \underbrace{\left(\frac{\partial L}{\partial \dot{q}_\mathsf{i}} - p_\mathsf{i}\right)^{\mathrm{T}}}_{=0 \text{ along motion}} \frac{\partial \dot{q}_\mathsf{i}}{\partial q_j}$$

- Along every motion with generalized momentum p_i

$$\frac{\partial R}{\partial q_\mathsf{n}} = \frac{\partial L}{\partial q_\mathsf{n}}, \ \frac{\partial R}{\partial \dot{q}_\mathsf{n}} = \frac{\partial L}{\partial \dot{q}_\mathsf{n}}, \ \frac{\partial R}{\partial t} = \frac{\partial L}{\partial t}, \ \frac{\partial R}{\partial p_\mathsf{i}} = -\dot{q}_\mathsf{i}$$

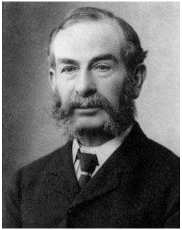
- Reduced equations for nonignorable coordinates

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_\mathsf{n}}\right) - \frac{\partial R}{\partial q_\mathsf{n}} = 0$$

- Quadrature for ignorable coordinates

$$\dot{q}_\mathsf{i} = -\frac{\partial R}{\partial p_\mathsf{i}}$$

Edward Routh



Edward Routh
1831-1907

- Dynamics
- Stability

Kepler's Problem Again

$$
\begin{aligned}
L &= \frac{1}{2}(r^2\dot{\theta}^2 + \dot{r}^2) + \frac{\mu}{r} \\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} \\
R &= -\frac{1}{2}\frac{p_\theta^2}{r^2} + \frac{1}{2}\dot{r}^2 + \frac{\mu}{r}
\end{aligned}
$$

$$
\ddot{r} - \frac{p_\theta^2}{r^3} + \frac{\mu}{r^2} = 0
$$

$$
\dot{\theta} = -\frac{\partial R}{\partial p_\theta} = \frac{p_\theta}{r^2}
$$

- Note: $R = T' - V',\ T' = \frac{1}{2}\dot{r}^2,\ V' = \frac{p_\theta^2}{2r^2} - \frac{\mu}{r}$
- For a given p_θ, $\dot{\theta}$ is a function of r
- V'=potential due to centrifugal force + gravity

Example: Spherical Pendulum

$$\begin{aligned} L &= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) - mgl(1-\cos\theta) \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta \text{ (angular momentum about vertical)} \\ R &= \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}\frac{p_\phi^2}{ml^2\sin^2\theta} - mgl(1-\cos\theta) \end{aligned}$$

$$ml^2\ddot{\theta} - \underbrace{\frac{p_\phi^2\cos\theta}{ml^2\sin^3\theta}}_{\text{centrifugal}} + \underbrace{mgl\sin\theta}_{\text{gravity}} = 0$$

Energy Integral

- Assume the system is *conservative*, that is
  - All applied forces are conservative
  - Lagrangian is independent of time
  - Velocity constraints are of the form $a_i^{\mathrm{T}}(q,t)\dot{q}=0$
    - ⋆ Implies position constraints on q are constant

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=\lambda_1 a_1+\cdots+\lambda_m a_m$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}^{\mathrm{T}}\dot{q}-L\right)=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}^{\mathrm{T}}\right)\dot{q}+\frac{\partial L}{\partial \dot{q}}^{\mathrm{T}}\ddot{q}-\frac{\partial L}{\partial \dot{q}}^{\mathrm{T}}\ddot{q}-\frac{\partial L}{\partial q}^{\mathrm{T}}\dot{q}-\frac{\partial L}{\partial t}$$

$$=\quad[\lambda_1 a_1+\cdots+\lambda_m a_m]^{\mathrm{T}}\dot{q}-\frac{\partial L}{\partial t}$$

$$=\quad 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}^{\mathrm{T}}\dot{q}-L\right)=-\frac{\partial L}{\partial t}=0$$

- $h(q,\dot{q})=\frac{\partial L}{\partial \dot{q}}^{\mathrm{T}}\dot{q}-L$, Jacobi integral, energy integral

Form of the Jacobi Integral

$$L = T_2 + T_1 + T_0 - V$$

$$T_2 = \frac{1}{2}\dot{q}^{\mathrm{T}} M \dot{q},\ T_1 = \left[\frac{\partial x}{\partial t}^{\mathrm{T}} J \frac{\partial x}{\partial q}\right]\dot{q},\ T_0 = \frac{1}{2}\frac{\partial x}{\partial t}^{\mathrm{T}} J \frac{\partial x}{\partial t}$$

$$\frac{\partial L}{\partial \dot{q}}^{\mathrm{T}} \dot{q} = \left[\dot{q}^{\mathrm{T}} M + \frac{\partial x}{\partial t}^{\mathrm{T}} J \frac{\partial x}{\partial q}\right]^{\mathrm{T}} \dot{q} = 2T_2 + T_1$$

$$\begin{aligned} h &= 2T_2 + T_1 - L \\ &= T_2 + (V - T_0) \\ &= T' + V' \end{aligned}$$

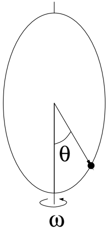
$T' = T_2 =$ Kinetic energy when all moving constraints/forces are held stationary

$V' = V - T_0 =$ Potential energy that includes effect of inertia forces due to moving constraints

Jacobi Integral and Total Energy

- Energy integral equals real energy if $T' = T, V' = V$, that is, $T_1 = T_0 = 0$
  - $T_0 = 0 \Longrightarrow \dfrac{\partial x}{\partial t} = 0$, transformation does not depend on time
- A system is called *natural* if $T = T_2$
- Fact: Total energy of a natural system is conserved if the Lagrangian is independent of time

Example: Particle on a Rotating Hoop



$$
\begin{aligned}
L &= \frac{1}{2}m\left(r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta + 2gr\cos\theta\right) \\
h &= \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}mr^2\omega^2\sin^2\theta - mgr\cos\theta \\
V' &= -mgr\cos\theta - \frac{1}{2}mr^2\omega^2\sin 2\theta \\
-\frac{\partial V'}{\partial\theta} &= \underbrace{-mgr\sin\theta}_{\text{gravity torque}} + \underbrace{mr^2\omega^2\sin\theta\cos\theta}_{\text{centrifugal torque}}
\end{aligned}
$$

Example: Reduced Kepler's Problem

$$R = \frac{1}{2}\dot{r}^2 - \frac{1}{2}\frac{p_\theta^2}{r^2} + \frac{\mu}{r}$$

$$h = \frac{\partial R}{\partial \dot{r}}\dot{r} - R = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{p_\theta^2}{r^2} - \frac{\mu}{r}$$

$$V' = \frac{1}{2}\frac{p_\theta^2}{r^2} - \frac{\mu}{r} = \text{potential due to centrifugal} + \text{gravity}$$

- Can solve reduced problem by quadratures

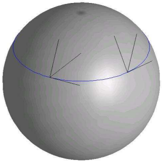
$$\dot{r} = \sqrt{2h - \frac{p_\theta^2}{r^2} + \frac{\mu}{r}}$$

  - Reduction by using ignorable coordinates, solutions by using energy integral

Ignorable Coordinates Revisited

- If one set of coordinates has an ignorable coordinates, would every other set have one too?
- Spherical pendulum: ϕ ignorable if, for every $\theta, \dot{\phi}, \dot{\theta}$ and ϕ_1, ϕ_2,

$$L(\phi_1, \theta, \dot{\phi}, \dot{\theta}) = L(\phi_2, \theta, \dot{\phi}, \dot{\theta}))$$



  - Lagrangian is invariant under rotations of position and velocity about the vertical axis
  - Not invariant under rotations about any other axis
  - Any other set of spherical coordinates will not have an ignorable coordinates
- System should continue to have an integral of motion in any coordinates!
- How to find integrals of motion when ignorable coordinates are not obvious?

Transformations

- Existence of ignorable coordinates related to invariance of L under some transformation of $q, \dot{q}$
- A *transformation* on the configuration space is an invertible function $h : \mathcal{Q} \to \mathcal{Q}$
  - Rotation about a given axis for a spherical pendulum
  - Rotation about symmetry axis for a particle on a cylinder
  - Rotation about center of attraction in Kepler's problem
  - Rotation about center of mass of a rigid body
  - Translation of the center of mass by a given vector v
- The set of all transformations on $\mathcal{Q}$ is a group $\mathcal{G}$
  - If h_1, h_2 aare transformations, then so are $h_1 \circ h_2$, h_1^{-1}
  - The identity map $\mathrm{id} : \mathcal{Q} \to \mathcal{Q}$ given by $\mathrm{id}(q) = q$ is a transformation

One-Parameter Group of Transformations

- A one-parameter group of transformations on $\mathcal{Q}$ is a map

$$h : \mathbb{R} \to \mathcal{G}$$

such that $h_{s1} \circ h_{s2} = h_{s1+s2},\ h_0 = \mathrm{id}$

- Example: Rotation about z-axis through angle s

Cartesian coordinates: $q = (x, y, z)$,

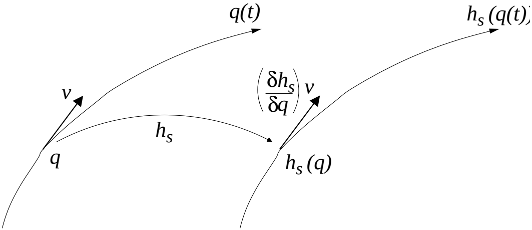
$$h_s(q) = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Spherical coordinates: $q = (r, \theta, \phi)$, $h_s(r, \theta, \phi) = (r, \theta, \phi + s)$

- Example: Translation along a vector w by amount s

$$h_s(q) = q + sw$$

Transformation of Velocities



- A velocity v at q_0 transforms to

$$\left.\frac{d}{dt}\right|_{t=0} h_s(q(t)) = \frac{\partial h_s}{\partial q}(q_0)v$$

where $q(t)$ is any curve satisfying $\dot{q}(0) = v, q(0) = q_0$

Translation: $h_s(q) = q + sw,\ h_s(q(t)) = q(t) + sw,\ \frac{\partial h_s}{\partial q}\dot{q}(0) = \dot{q}(0)$

Rotation:

$$\left.\frac{d}{dt}\right|_{t=0} h_s(r(t), \theta(t), \phi(t)) = \left.\frac{d}{dt}\right|_{t=0} (r(t), \theta(t), \phi(t) + s) = (\dot{r}(0), \dot{\theta}(0), \dot{\phi}(0))$$

Invariance Under a One-Parameter Group

- A Lagrangian L is *invariant* under the one-parameter group of transformations h_s if

$$L(q,v,t) = L\left(h_s(q), \frac{\partial h_s}{\partial q}(q)v, t\right)$$

  for every s, q, v, t
  - L has same value at all $(q,\dot{q})$ obtained by transforming the original $(q,\dot{q})$
- h_s is a one-parameter group of *symmetries*
- Example: Spherical pendulum

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$$

$$h_s(q) = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \frac{\partial h_s}{\partial q}(q)\dot{q} = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

  - Check: L is invariant under h_s
- Example: Particle on a sphere, no gravity. L invariant under all rotations

Consequences of Invariance

- *Fact*: If $q(t)$ is a motion of the system, then so is $h_s(q(t))$ for every s
  - $h_s(q(t))$ satisfies Lagrange's equation if $q(t)$ does
  - Assuming no non-conservative forces, holonomic constraints
  - A transformed motion is also a motion
- Noether's Theorem: If L is invariant under the one-parameter group of transformations h_s, then $p(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, t)^{\mathrm{T}} \left.\frac{d}{ds}\right|_{s=0} h_s(q)$ is a first integral
  - $p(q, \dot{q}, t)=$ generalized momentum along the direction in which h_s tends to change the configuration

Proof of Noether's Theorem

- Let L be invariant under h_s, and $q(t)$ be a motion. For all s, t

$$L\big(h_s(q(t)), \frac{\partial}{\partial t} h_s(q(t)), t\big) = L\big(q(t), \dot{q}(t), t\big)$$

$$\frac{\partial L}{\partial q}\left(h_s(q(t)), \frac{\partial}{\partial t} h_s(q(t)), t\right)^{\mathrm{T}} \frac{\partial}{\partial s} h_s(q(t)) + \frac{\partial L}{\partial \dot{q}}\left(h_s(q(t)), \frac{\partial}{\partial t} h_s(q(t)), t\right)^{\mathrm{T}} \frac{\partial}{\partial s}\left(\frac{\partial}{\partial t} h_s(q(t))\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\big(h_s(q(t)), \frac{\partial}{\partial t} h_s(q(t)), t\big)\right)^{\mathrm{T}} \frac{\partial}{\partial s} h_s(q(t)) + \frac{\partial L}{\partial \dot{q}}\left(h_s(q(t)), \frac{\partial}{\partial t} h_s(q(t)), t\right)^{\mathrm{T}} \frac{\partial}{\partial s}\left(\frac{\partial}{\partial t} h_s(q(t))\right) = 0$$

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{q}}\left(h_s(q(t)), \frac{\partial}{\partial t} h_s(q(t)), t\right)^{\mathrm{T}} \frac{\partial}{\partial s} h_s(q(t))\right] = 0$$

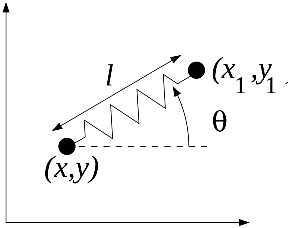
- Put $s = 0$ $\quad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\left(q(t), \dot{q}(t), t\right)^{\mathrm{T}} \frac{\partial}{\partial s}\Big|_{s=0} h_s(q(t))\right] = 0$

Emmy Noether

Emmy Noether
1882-1935

- Algebra
  - Theory of Rings

Noether's Theorem: An Example



$$T = \frac{1}{2}m\left[2\dot{x}^2 + 2\dot{y}^2 + \dot{l}^2 + \dot{l}^2\dot{\theta}^2 + \right.$$
$$\left. 2\dot{l}(\dot{x}\cos\theta + \dot{y}\sin\theta) + 2l\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta)\right]$$

$$V = \tfrac{1}{2}k(l - l_0)^2$$

- L is invariant under translations along x
  $$h_s(x, y, l, \theta) = \begin{bmatrix} x+s & y & l & \theta \end{bmatrix}^{\mathrm{T}}$$
  $$\left.\frac{d}{ds}\right|_{s=0} h_s(x, y, l, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

  $$\begin{aligned} p_1 &= \begin{bmatrix} \frac{\partial L}{\partial \dot{x}} & \frac{\partial L}{\partial \dot{y}} & \frac{\partial L}{\partial \dot{l}} & \frac{\partial L}{\partial \dot{\theta}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \\ &= \frac{\partial L}{\partial \dot{x}} = 2m\dot{x} + m\dot{l}\cos\theta - ml\dot{\theta}\sin\theta = m\dot{x} + m\dot{x}_1 = x \text{ linear momentum} \end{aligned}$$

- L is invariant under transformation along $y \Rightarrow y$ linear momentum is conserved

Example (cont'd)

- Rotation about the origin

$$h_s(x, y, l, \theta) = \begin{bmatrix} x\cos s - y\sin s & x\sin s + y\cos s & \theta + s & l \end{bmatrix}^{\mathrm{T}}$$

$$\frac{\partial}{\partial t} h_s(x(t), y(t), l(t), \theta(t)) = \begin{bmatrix} \dot{x}\cos s - \dot{y}\sin s & \dot{x}\sin s + \dot{y}\cos s & \dot{\theta} & \dot{l} \end{bmatrix}^{\mathrm{T}}$$

- L is invariant under rotations (check)

$$\left.\frac{d}{ds}\right|_{s=0} h_s(q) = \begin{bmatrix} -y & x & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} p_3 = \left(\frac{\partial L}{\partial t}\right)^{\mathrm{T}} \left.\frac{d}{ds}\right|_{s=0} h_s(q) &= m(x + l\cos\theta)(\dot{y} + \dot{l}\sin\theta + l\dot{\theta}\cos\theta) \\ &\quad - m(y + l\sin\theta)(\dot{x} + \dot{l}\cos\theta - l\dot{\theta}\sin\theta) \\ &\quad + m(x\dot{y} - y\dot{x}) \\ &= \mathbf{k}\cdot[m(\mathbf{r}_1 \times \dot{\mathbf{r}}_1) + m(\mathbf{r}_2 \times \dot{\mathbf{r}}_2)] \\ &= \text{angular momentum about origin} \end{aligned}$$