Spaceflight Dynamics

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Outline

Introduction

- Space engineering
  - Supports astronomy, astrophysics, space sciences, telecommunications, military, meteorology
- Through spacecraft such as
  - Interplanetary spacecraft
  - Earth satellites
    - Unmanned satellites
    - Manned space stations
  - Reusable space vehicles
- This course — earth satellites

Where to put them?

- Orbit dictated by mission
- Orbit described in terms of shape, size and orientation
- Orbit depends on position and velocity at the start of orbital motion
- Orbital mechanics
    - Description and prediction of orbital motion

Aristotle 384BC-322BC

Ptolemy 85AD-165AD

Copernicus 1473-1543

Galileo 1564-1642

Kepler 1571-1630

Newton 1643-1727

What do they do up there?

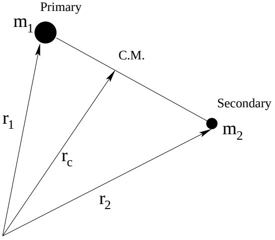
- Attitude dynamics (rotational motion)
  - Description
    - Variables
    - Equations of motion
    - Solutions
- Attitude control

How to put them there?

- Satellites injected by launch vehicles
- Initial conditions for orbital motion decided by burnout position and velocity
- Rocket performance
  - Limited by structural mass
  - Leads to staging
- Rocket trajectories
  - Predict burnout conditions

Two-Body Problem

- Motion of two bodies moving under mutual gravitational acceleration



$$m_1\ddot{\mathbf{r}}_1 = -\frac{Gm_1m_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2)$$

$$m_2\ddot{\mathbf{r}}_2 = -\frac{Gm_1m_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_2-\mathbf{r}_1)$$

Translation of Center of Mass

- Six degrees of freedom
  - Three for motion of center of mass
  - Three for relative motion

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathrm{c}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0$$

- Center of mass moves along a straight line with uniform velocity

Relative Motion

- In terms of displacement vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ of secondary relative to primary

$$m_1 \times \text{second equation} - m_2 \times \text{first equation} \Longrightarrow$$

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{|\mathbf{r}^3|}, \quad \mu = G(m_1 + m_2)$$

- Central force motion with inverse square attraction
- Cannot be derived by using Newton's law directly

Energy Integral

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\frac{\mu}{r^3}(\mathbf{r} \cdot \dot{\mathbf{r}}), \quad r = |\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}}$$

$$\text{LHS} = \frac{d}{dt}\left(\frac{1}{2}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\right) = \frac{d}{dt}\left(\frac{1}{2}v^2\right)$$

$$\text{RHS} = -\frac{\mu}{r^3}\frac{d}{dt}\left(\frac{1}{2}(\mathbf{r} \cdot \mathbf{r})\right) = -\frac{1}{2}\frac{\mu}{r^3}\frac{d}{dt}(r^2) = -\frac{\mu}{r^3}r\dot{r} = \frac{d}{dt}\left(\frac{\mu}{r}\right)$$

$$\therefore \ \frac{d}{dt}\underbrace{\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right)}_{\mathcal{E}} = 0$$

- Specific energy $\mathcal{E} = \frac{1}{2}v^2 - \mu r^{-1} =$ constant

  Note: $\mathcal{E} \neq$ total mechanical energy of the two-body system

Conclusions from the Energy Integral

$$v = \sqrt{2\mathcal{E} + 2\frac{\mu}{r}}$$

- If $\mathcal{E} < 0$, then $v = 0$ at $r = -\mu^{-1}\mathcal{E}$
  - Satellite falls back, orbit is bounded
- If $\mathcal{E} \geq 0$, satellite can be in motion at any distance
  - Satellite escapes ??
- Escape speed at distance r

$$v_{\text{esc}} \stackrel{\text{def}}{=} \sqrt{\frac{2\mu}{r}}$$

Note: Escape verified as possible but not guaranteed

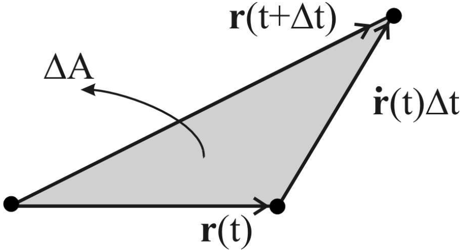
Angular Momentum

- Specific angular momentum $\mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}}$

$$\begin{aligned} \dot{\mathbf{H}} &= \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0 \\ \mathbf{H} &= \text{Constant along the orbit} \end{aligned}$$

- $\mathbf{r}$ and $\dot{\mathbf{r}}$ lie in a fixed plane perpendicular to the constant $\mathbf{H}$
- Orbit lies in a plane
  - Only uses the fact that $\mathbf{H}$ has a constant direction

Areal Rate



- Area swept out by the radius vector in a small time increment Δt

$$\Delta A = \frac{1}{2}\,|\mathbf{r}(t) \times \mathbf{r}(t+\Delta t)|$$
$$= \frac{1}{2}\,\Delta t\,|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)| = \frac{1}{2}\,\Delta t\,|\mathbf{H}|$$

$$\frac{dA}{dt} = \frac{1}{2}\,|\mathbf{H}| = \frac{1}{2}\,H = \text{constant}$$

Areal Rate

- Kepler's law of areas: Radius vector sweeps out equal areas in equal interval of time

- So far: Speed as function of radius

  Planar nature of orbit

  Motion along the orbit

- Next: Shape

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Eccentricity Vector

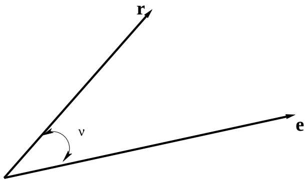
- Define eccentricity vector

$$\mathbf{e} \stackrel{\text{def}}{=} \mu^{-1}(\dot{\mathbf{r}} \times \mathbf{H}) - r^{-1}\mathbf{r}$$

  - Lies in the plane of motion
  - $\dot{\mathbf{e}} = 0$ along motion

- Define *true anomaly* ν to be angle between $\mathbf{r}$ and $\mathbf{e}$
- *Eccentricity* $e \stackrel{\text{def}}{=} \sqrt{\mathbf{e} \cdot \mathbf{e}}$

$$\begin{aligned} \mu r e \cos\nu &= \mu \mathbf{r} \cdot \mathbf{e} = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{H}) \mu r \\ &= H^2 - \mu r \\ \therefore \ r &= \frac{H^2/\mu}{1 + e\cos\nu} \end{aligned}$$

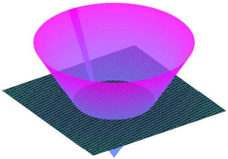


Nature of the Orbit

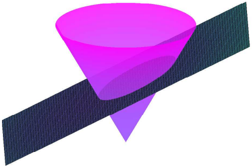
$$r = \frac{H^2/\mu}{1 + e\cos\nu}$$

- Polar equation of the orbit with
  - $\mathbf{e}$ as the positive x-axis
  - Primary body as the origin
- Orbit bounded if and only if $e < 1$
- Also the polar equation of a conic section of eccentricity e with origin at its focus
- Kepler's law of orbits: Orbit is a conic section with focus at its primary
- Conic section: curve of intersection between a right circular cone and a plane

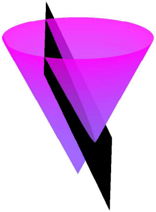
Conic Sections



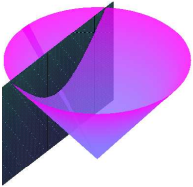
Circle



Ellipse



Parabola



Hyperbola

Shape and Size of the Orbit

- Shape determined by the eccentricity e

$$e = \sqrt{\frac{2H^2\mathcal{E}}{\mu^2} + 1}$$

  - $e = 0 \;\Rightarrow$ circular orbit
  - $0 < e < 1 \;\Rightarrow$ elliptic orbit
  - $e = 1 \;\Rightarrow$ parabolic orbit
  - $e > 1 \;\Rightarrow$ hyperbolic orbit

- Size determined by the semilatus rectum H^2/μ

Circular Orbits

- Zero eccentricity $\Rightarrow$ $r = H^2/\mu =$ constant
- For a circular orbit, $H = rv$

$$\Longrightarrow \text{orbital speed at radius } r \quad v = \sqrt{\frac{\mu}{r}}$$

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{1}{2}\frac{\mu}{r} < 0$$

Elliptic Orbits

- $0 < e < 1$, orbit is elliptical with one focus at the primary
- Periapsis (perigee/perihelion) point of closest approach at $\nu = 0$

$$r_p = \frac{H^2/\mu}{1+e}$$

- Apoapsis (apogee/aphelion) farthest point from the primary at $\nu = \pi$

$$r_a = \frac{H^2/\mu}{1-e}$$

- Semimajor axis

$$a = \frac{r_p + r_a}{2} = \frac{H^2/\mu}{1-e^2}$$

$$H = \sqrt{\mu a(1-e^2)}$$

- Speed at periapsis

$$v_p = \frac{H}{r_p} = \sqrt{\frac{\mu}{a}\frac{(1+e)}{(1-e)}}$$

- Total specific energy

$$\mathcal{E} = \frac{1}{2}v_p^2 - \frac{\mu}{r_p} = -\frac{\mu}{2a}$$

Geometrical and Mechanical Description

$$H = \sqrt{\mu a(1-e^2)} \qquad a = -\frac{\mu}{2\mathcal{E}}$$

$$\mathcal{E} = -\frac{\mu}{2a} \qquad e = \sqrt{\frac{2H^2\mathcal{E}}{\mu^2} + 1}$$

$$r_p = a(1-e) \qquad r_a = a(1+e)$$

$$v_p = \sqrt{\frac{\mu}{a}\frac{1+e}{1-e}} \qquad v_a = \sqrt{\frac{\mu}{a}\frac{(1-e)}{(1+e)}}$$

Seminor axis $\quad b = a\sqrt{1-e^2}$

Parabolic Orbits

- $e = 1,\ r = \dfrac{H^2/\mu}{1 + \cos\nu}$
- Periapsis distance $r_p = \dfrac{H^2}{2\mu},\ v_p = \dfrac{H}{r_p} = \dfrac{2\mu}{H}$

$$\mathcal{E} = \frac{1}{2}v_p^2 - \frac{\mu}{r_p} = 0$$

$\therefore\ v = \sqrt{\dfrac{2\mu}{r}} \to 0$ as $r \to \infty$

  - Just enough energy to reach ∞ at rest

- $v_{\mathrm{esc}} = \sqrt{\dfrac{2\mu}{r}}$ is sufficient for escape. $v \geq v_{\mathrm{esc}}$ guarantees escape

Hyperbolic Orbits

- $e > 1$, orbit is one branch of a hyperbola with its focus at the primary

$$r \to \infty \text{ as } v \to v_\infty \stackrel{\text{def}}{=} \pi - \cos^{-1} 1/e$$

$$\text{speed } v = \sqrt{2(\mathcal{E} + \frac{\mu}{r})} \to \text{hyperbolic excess velocity } v_\infty \stackrel{\text{def}}{=} \sqrt{2\mathcal{E}} \text{ as } r \to \infty$$

- Periapsis distance

$$r_p = \frac{H^2/\mu}{1+e} \qquad v_p = \frac{H}{r_p} = \frac{\mu(1+e)}{H}$$

$$\mathcal{E} = \frac{\mu^2(e^2-1)}{2H^2} \qquad v_\infty = \frac{\mu}{H}\sqrt{e^2-1}$$

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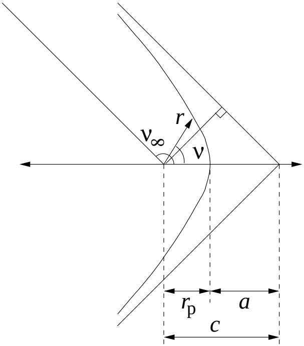
Geometric Description of Hyperbolic Orbits



$$H = v_{\infty} c \sin \nu_{\infty} = c \frac{\mu}{H} \sqrt{e^2 - 1} \left(\frac{\sqrt{e^2 - 1}}{e} \right)$$

$$c = \frac{H^2}{\mu} \frac{e}{e^2 - 1}$$

$$\text{Semimajor axis } a = c - r_p = \frac{H^2}{\mu(e^2 - 1)} = \frac{c}{e}$$

Geometric versus Mechanical Description

$$H = \sqrt{\mu a(e^2 - 1)} \qquad a = \frac{\mu}{2\mathcal{E}}$$

$$\mathcal{E} = \frac{\mu}{2a} \qquad e = \sqrt{\frac{2H^2\mathcal{E}}{\mu^2} + 1}$$

$$r_p = a(e-1)$$

$$v_p = \sqrt{\frac{\mu}{a}\frac{(e+1)}{(e-1)}}$$

$$v_\infty = \sqrt{\frac{\mu}{a}}$$

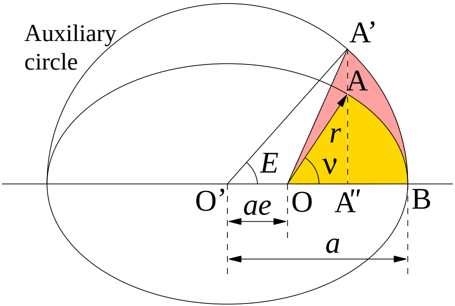
Motion Along an Elliptic Orbit: Orbital Period

- Total area of orbit $= \pi ab = \pi a^2\sqrt{1-e^2}$
- Areal rate $\dfrac{dA}{dt} = \dfrac{H}{2} = \dfrac{1}{2}\sqrt{\mu a(1-e^2)}$
  - Orbital period $T = 2\pi\sqrt{\dfrac{a^3}{\mu}}$
- Kepler's law of periods

$$(\text{period})^2 \propto (\text{semimajor axis})^3$$

Motion along an Elliptic Orbit: Kepler's Equation

- Need position along the orbit as function of time
- Use Kepler's law of areas — need area of a sector of an ellipse



$$\text{Area(OAB)} = \frac{b}{a}\text{Area(OA}'\text{B)} = \frac{b}{a}\left[\text{Area(O}'\text{A}'\text{B)} - \text{Area(O}'\text{A}'\text{O)}\right]$$

$$\text{Area(O}'\text{A}'\text{B)} = \tfrac{1}{2}a^2E, \ \text{Area(O}'\text{A}'\text{O)} = \tfrac{1}{2}(ae)a\sin E = \tfrac{1}{2}a^2 e\sin E$$

$$\text{Area(OAB)} = \frac{1}{2}ab(E - e\sin E)$$

Kepler's Equation

- Let t_p be the instant of periapsis passage

$$\left.\frac{\text{Area(OAB)}}{t - t_p} = \frac{1}{2}H \quad \right\} \text{ Law of areas}$$

$$E - e\sin E = \frac{H(t-t_p)}{ab} = \sqrt{\frac{\mu}{a^3}}(t-t_p)$$

- Define *mean motion*

$$n \stackrel{\text{def}}{=} \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

- Kepler's equation:

$$E - e\sin E = \underbrace{n(t-t_p)}_{\text{Mean anomaly } M}$$

True and Eccentric Anomalies

- Need to relate E and ν

$$\begin{aligned} \mathrm{O'A''} &= \mathrm{OO'} + \mathrm{OA''} \\ a\cos E &= ae + r\cos\nu \end{aligned}$$

- Use polar equation of the orbit

$$\cos E = \frac{(e+\cos\nu)}{(1+e\cos\nu)}$$

$$2\sin^2(E/2) = 1-\cos E = \frac{(1-e)2\sin^2(\nu/2)}{(1+e\cos\nu)}$$

$$2\cos^2(E/2) = 1+\cos E = \frac{(1+e)2\cos^2(\nu/2)}{(1+e\cos\nu)}$$

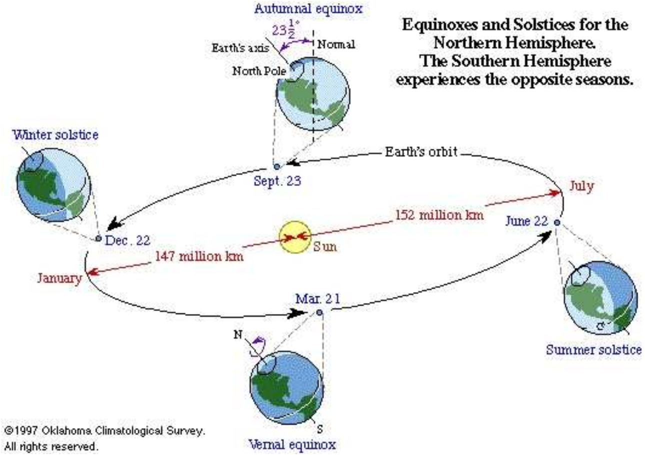
$$\tan(E/2) = \sqrt{\frac{(1-e)}{(1+e)}}\tan(\nu/2)$$

- Use along with Kepler's equation to find ν as function of time

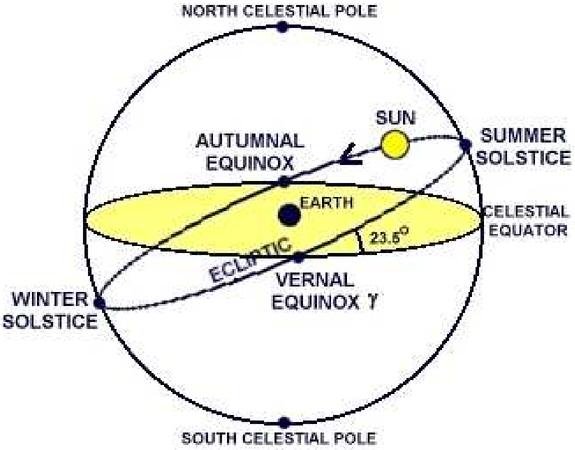
Geocentric Frame

- Need to describe orientation of the orbit or position of the satellite
  - With respect to an earth/inertial frame
  - Using quantities that help to visualize the orbit
- Define a non-rotating geocentric frame with
  - Origin at earth's center
  - Axes directions fixed with respect to solar system
- Z axis along earth's axis of rotation pointing north
  - Precesses with a period of $25,800$ years
  - Nutates with an amplitude $9''$ and period 18.6 years
- X axis along the line of intersection of earth's orbital plane (ecliptic) and earth's equatorial plane
  - Along line joining the equinoxes, pointing along the vernal equinox
  - Along the first point of Aries

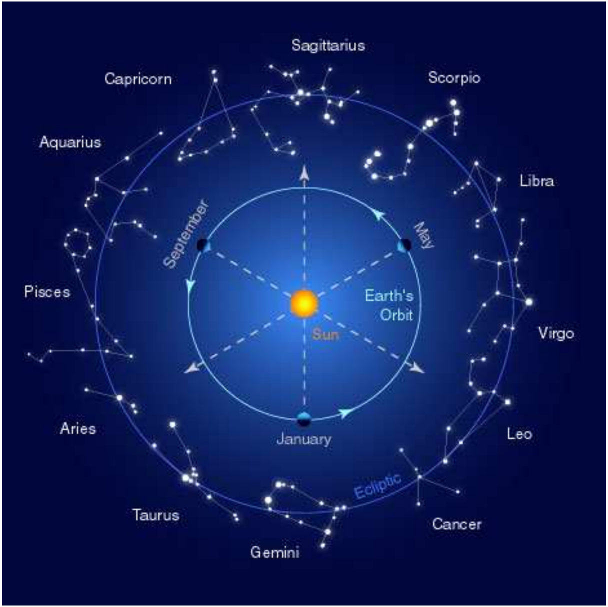
The Ecliptic



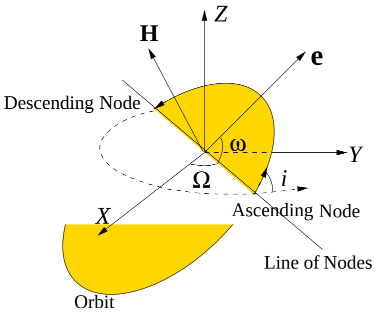
The Ecliptic: View from Earth



First Point of Aries



Orientation of the Orbit: Right Ascension



- Ascending node — point where the orbit crosses equatorial plane from S to N
- *Right ascension* of ascending node Ω
  - Eastward from the X axis to the ascending node

$$0 \leq \Omega < 2\pi$$

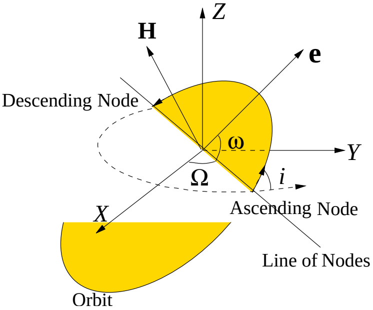
Orientation of the Orbit: Inclination

- Inclination of the orbit i
  - Measured at the ascending node between east and direction of motion

$$0 \leq i < \pi$$

    - ⋆ $i < 90^\circ$ — prograde orbit, orbital motion in the same direction as earth's rotation
    - ⋆ $i > 90^\circ$ — retrograde orbit
    - ⋆ $i \simeq 90^\circ$ — polar orbit
    - ⋆ $i = 0^\circ$ — equatorial orbit
    - ⋆ Inclination determines north and south limits of visibility

Orientation of the Orbit: Argument of Perigee



- Argument of perigee ω — measured in the orbital plane from the ascending node along the motion $0 \leq \omega < 2\pi$
- $\Omega,\ i,\ \omega$ describe the orientation of the orbit with respect to the geocentric frame
- In addition, a determines the size, e the shape
- Six classical orbital elements $a,\ e,\ i,\ \omega,\ \Omega,\ t_{\mathsf{p}}$

Determination of Classical Elements from Initial Conditions

- Given $\mathbf{r}$ and $\mathbf{v} = \dot{\mathbf{r}}$
  - Compute $\mathcal{E}$, $\mathbf{H}$, H, $\mathbf{e}$, e
- Line of nodes is perpendicular to $\mathbf{H}$ and $\mathbf{k}$
  - Unit vector along the line of nodes (pointing to the ascending node)

$$\mathbf{n} = |\mathbf{k} \times \mathbf{H}|^{-1}(\mathbf{k} \times \mathbf{H})$$

  - $\Omega \in [0,\ 2\pi)$ from $\quad \mathbf{n} = \cos(\Omega)\mathbf{i} + \sin(\Omega)\mathbf{j}$
- Compute $i \in [0,\ \pi]$ from $\quad \cos i = \dfrac{\mathbf{k} \cdot \mathbf{H}}{H}$
- Compute $\omega \in [0,\ 2\pi)$, the angle between $\mathbf{n}$ and $\mathbf{e}$, by $\cos\omega = \dfrac{\mathbf{n} \cdot \mathbf{e}}{e}$

$$\begin{aligned} \omega &= \cos^{-1}\frac{\mathbf{n}\cdot\mathbf{e}}{e}, \qquad \mathbf{e}\cdot\mathbf{k} \geq 0 \\ &= 2\pi - \cos^{-1}\frac{\mathbf{n}\cdot\mathbf{e}}{e}, \quad \mathbf{e}\cdot\mathbf{k} < 0 \end{aligned}$$

Determination of Classical Elements (continued)

- To find t_{p}, first find initial anomaly ν from $\cos\nu = \dfrac{\mathbf{e}\cdot\mathbf{r}}{er}$

$$\begin{aligned} \nu &= \cos^{-1}\frac{\mathbf{e}\cdot\mathbf{r}}{er}, \quad \mathbf{v}\cdot\mathbf{e}\le 0 \quad \text{(satellite traveling from perigee to apogee)} \\ &= 2\pi - \cos^{-1}\frac{\mathbf{e}\cdot\mathbf{r}}{er}, \quad \mathbf{v}\cdot\mathbf{e} > 0 \quad \text{(satellite traveling from apogee to perigee)} \end{aligned}$$

- Compute initial eccentric anomaly

$$\tan(E/2) = \sqrt{\frac{(1-e)}{(1+e)}}\tan(\nu/2)$$

- Compute t_{p} from Kepler's equation

$$\sqrt{\frac{\mu}{a^3}}(t_0 - t_{\mathsf{p}}) = E - e\sin E$$

Determination of Position and Velocity

- Given $t, \; a, \; e, \; i, \; \Omega, \; \omega, \; t_{\mathsf{p}}$
- Compute eccentric anomaly at t

$$E - e \sin E = \sqrt{\frac{\mu}{a^3}}(t - t_{\mathsf{p}})$$

- Compute true anomaly at t

$$\sqrt{\frac{(1-e)}{(1+e)}} \tan(\nu/2) = \tan(E/2)$$

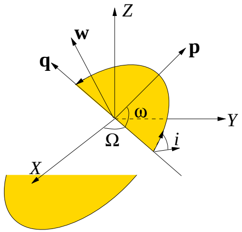
- Compute geocentric distance at t

$$r = \frac{a(1-e^2)}{1 + e\cos\nu}$$

  - Position in the orbital plane determined
  - Need to transform to geocentric coordinates

Position in Perifocal Frame

- Introduce a perifocal coordinate system, origin at earth's center and unit vectors
  - $\mathbf{p}$ pointing to the perigee
  - $\mathbf{q}$ along the position $\nu = 90^\circ$
  - $\mathbf{w}$ orthogonal to the orbital frame such that $\mathbf{p} \times \mathbf{q} = \mathbf{w}$



$$\mathbf{r} = r\cos\nu\ \mathbf{p} + r\sin\nu\ \mathbf{q}$$

Velocity in Perifocal Frame

$$\dot{\mathbf{r}} = (\dot{r}\cos\nu - r\dot{\nu}\sin\nu)\mathbf{p} + (\dot{r}\sin\nu + r\dot{\nu}\cos\nu)\mathbf{q}$$

- To find $r\dot{\nu}$, note $\mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}} = r^2\dot{\nu}\mathbf{w}$

$$r\dot{\nu} = \frac{H}{r} = \sqrt{\frac{\mu}{a(1-e^2)}}(1 + e\cos\nu)$$

- To find $\dot{r}$, differentiate polar equation

$$\dot{r} = \sqrt{\frac{\mu}{a(1-e^2)}}e\sin\nu$$

- Perifocal components of velocity

$$\dot{\mathbf{r}} = \sqrt{\frac{\mu}{a(1-e^2)}}[-\sin\nu\ \mathbf{p} + (e + \cos\nu)\ \mathbf{q}]$$

- Need to transform to geocentric frame

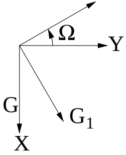
Transformation to Geocentric Frame

- To obtain the transformation, perform a sequence of 3 rotations on G to get P
  - Rotate G about Z through Ω to get G_1
  - Rotate G_1 about X through i to get G_2
  - Rotate G_2 about Z through ω to get P
- For any vector $\mathbf{r}$

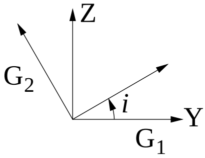
$$\begin{aligned} \mathbf{r}_G &= R_1(\Omega)(\mathbf{r}_{G_1}) \\ \mathbf{r}_{G_1} &= R_2(i)(\mathbf{r}_{G_2}) \\ \mathbf{r}_{G_2} &= R_3(\omega)(\mathbf{r}_P) \end{aligned}$$

$$\mathbf{r}_G = R_1(\Omega)R_2(i)R_3(\omega)\mathbf{r}_P$$

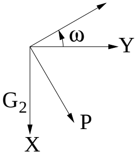
Transformation Matrices



$$R_1(\Omega) = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_1(i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}$$



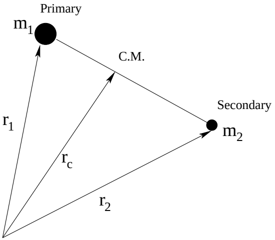
$$R_1(\omega) = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Geocentric Components of Position and Velocity

$$\mathbf{r}_G = R_1(\Omega)R_2(i)R_3(\omega)\begin{bmatrix} r\cos\nu \\ r\sin\nu \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{r}}_G = \sqrt{\frac{\mu}{a(1-e^2)}}R_1(\Omega)R_2(i)R_3(\omega)\begin{bmatrix} -\sin\nu \\ e+\cos\nu \\ 0 \end{bmatrix}$$

Complete Solution of the Two-Body Problem



$$\mathbf{r}_2 - \mathbf{r}_c = \frac{m_1}{m_1 + m_2}\mathbf{r}$$
$$\mathbf{r}_1 - \mathbf{r}_c = -\frac{m_2}{m_1 + m_2}\mathbf{r}$$

- Each body moves along a conic section with focus at the center of mass

Satellite Tracking and Orbit Determination

- Predicting orbit from measured position and velocity data
- Position and velocity known only at injection point from launch vehicle INS
- Optical tracking
  - Each observation yields right ascension and declination, no range information
  - Three observations required to determine orbit
  - Observations made from rotating, translating earth
  - Approximate method by Laplace, exact method by Gauss
- Radar tracking for low-earth satellites
  - Azimuth, elevation, range in each observation
  - Some method interpolate between closely spaced observations, differentiate to get velocity
  - Other use two position measurement with elapsed time
- Range–range-rate tracking for deep space craft
  - Range-rate measured by using Doppler shift
  - No angular information available

Errors in Orbit Determination

- Measurement errors lead to errors in estimated orbital parameters
- Errors between actual position and estimated position grows with time
  - Example: error in period
  - Need for improving accuracy by making new observations and updating the orbit
  - Need for correcting the orbit
- Body of observations increases with time
  - Use all data rather than the minimum amount required
  - Best fit — method of least squares

Carl Friedrich Gauss

Carl Friedrich Gauss
1777-1855

- Number theory
- Astronomy
- Statistics
- Analysis
- Differential geometry
- Geodesy
- Geomagnetism

Orbital Maneuvers

- Needed to transfer a geostationary satellite from its low earth parking orbit to its final high altitude geostationary orbit
- Needed to correct changes in orbital elements due to perturbing forces
- Impulsive thrust maneuvers
  - Velocity changes instantaneously without change in position
  - Thrust duration (burn times) small compared to orbital period (coast time)
- Hohmann transfer between two coplanar circular orbits

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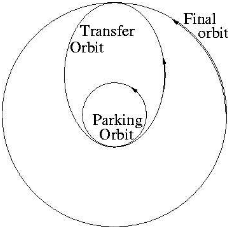
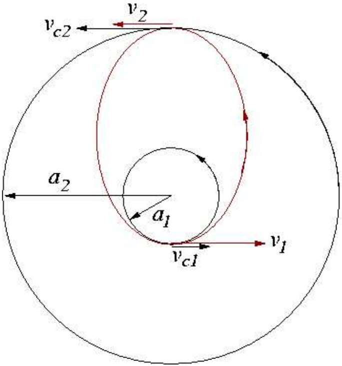


Hohmann Transfer



- Two impulsive maneuvers
  - Apogee boost: increase speed from v_{c_1} to v_1 so that the satellite enters an elliptical transfer orbit with apogee on the final orbit
  - Circularization: increase speed at the apogee of the transfer orbit to enter the final circular orbit

Hohmann Transfer (cont'd)

- Circular orbits

$$v_{c_1} = \sqrt{\frac{\mu}{a_1}}, \quad v_{c_2} = \sqrt{\frac{\mu}{a_2}}$$

- Transfer orbit

$$a = \frac{a_1 + a_2}{2}, \quad v_1 = \sqrt{2\left(\frac{\mu}{a_1} - \frac{\mu}{a_1 + a_2}\right)}, \quad v_2 = \sqrt{2\left(\frac{\mu}{a_2} - \frac{\mu}{a_1 + a_2}\right)}$$

- Impulse magnitudes

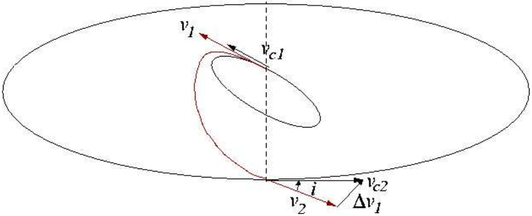
$$\begin{aligned} \Delta v_1 &= v_1 - v_{c_1} \\ \Delta v_2 &= v_{c_2} - v_2 \end{aligned}$$

- Minimum duration between maneuvers

$$= \frac{T}{2} = \pi\sqrt{\frac{a^3}{\mu}}$$

Inclination Change Maneuver

- Needed if geostationary satellite is not launched from the equator
- Combined with one of the maneuvers (usually second) of the Hohmann transfer
  - To calculate magnitude and direction of the impulse required



$$|\Delta \mathbf{v}_2|^2 = {v_{c_2}}^2 + {v_2}^2 - 2 v_{c_2} v_2 \cos i$$

Coordinate Transformation

- View classical elements as new coordinates

$$x \stackrel{\text{def}}{=} \begin{bmatrix} r_x & r_y & r_z & v_x & v_y & v_z \end{bmatrix}^{\mathrm{T}} = \Phi\left(a,\ e,\ i,\ \Omega,\ w,\ M\right)$$

- Inverse transformation known

$$\begin{bmatrix} a & e & i & \Omega & w & M \end{bmatrix}^{\mathrm{T}} = \Phi^{-1}(x)$$

- Equations of motion in the two-body problem

$$\dot{x} = \begin{bmatrix} \dot{r_x} & \dot{r_y} & \dot{r_z} & \dot{v_x} & \dot{v_y} & \dot{v_z} \end{bmatrix}^{\mathrm{T}} = f\left(r_x,\ r_y,\ r_z,\ v_x,\ v_y,\ v_z\right) = f(x)$$

- Use transformation to write equation of motion in terms of classical elements

$$\begin{bmatrix} \dot{a} & \dot{e} & \dot{i} & \dot{\Omega} & \dot{w} & \dot{M} \end{bmatrix}^{\mathrm{T}} = \frac{\partial \Phi}{\partial x}^{-1} f\left(\Phi\left(a,\ e,\ i,\ \Omega,\ w,\ M\right)\right)$$

- In the two-body problem

$$\begin{bmatrix} \dot{a} & \dot{e} & \dot{i} & \dot{\Omega} & \dot{w} & \dot{M} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}^{\mathrm{T}}$$

Perturbation Forces

- Inhomogeneity and oblateness of earth
- Third body gravitational influence, eg. sun, moon
- Solar wind
- Solar radiation pressure
- Atmospheric drag in low earth orbit

$$\dot{x} = f(x) + \underbrace{p(x,t)}_{\text{perturbation}}$$

$$\begin{bmatrix}\dot{a} & \dot{e} & \dot{i} & \dot{\Omega} & \dot{w} & \dot{M}\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & n\end{bmatrix}^{\mathrm{T}} + \text{perturbation}$$

- Trajectory no longer a conic section
  - Can be thought of as path traced by a point on an ellipse that is *osculating*, that is, changing shape, size and orientation

$$x(t) = \Phi\left(a(t),\ e(t),\ i(t),\ \Omega(t),\ w(t),\ M(t)\right)$$

Gauss's Planetary Equation

- Resolve perturbation force along perifocal frame

$$\text{Perturbation force} = P\mathbf{p} + Q\mathbf{q} + W\mathbf{w}$$

$$\begin{aligned}
\dot{a} &= \frac{2}{n\sqrt{1-e^2}}\left[eP\sin\nu + (1+e\cos\nu)\,Q\right] \\
\dot{e} &= \frac{\sqrt{1-e^2}}{na}\left[P\sin\nu + (\cos E + \cos\nu)Q\right] \\
\dot{i} &= \frac{1}{na\sqrt{1-e^2}}\frac{rW}{a}\cos(\nu+\omega) \\
\dot{\Omega} &= \frac{1}{na\sqrt{1-e^2}}\frac{rW}{a}\frac{\sin(\nu+\omega)}{\sin i} \\
\dot{\omega} &= \\
\dot{M} &=
\end{aligned}$$

Earth Inhomogeneity and Oblateness

- Gravitational potential due to earth in spherical coordinates

$$U(r,\lambda,\phi) = -\frac{\mu}{r} + \underbrace{B(r,\lambda,\phi)}_{\text{perturbation}}$$

$$B(r,\lambda,\phi) = \frac{\mu}{r}\left\{\sum_{n=2}^{\infty}\left[\underbrace{\left(\frac{R_\mathsf{e}}{r}\right)^n J_n P_n(\sin\lambda)}_{\text{oblateness}} + \sum_{m=1}^{n} J_{mn}\left(\frac{R_\mathsf{e}}{r}\right)^n \underbrace{(C_{nm}\cos m\phi + S_{nm}\sin m\phi)}_{\text{asymmetry}} P_{nm}(\sin\lambda)\right]\right\}$$

- R_e = mean equatorial radius
- P_n = Legendre polynomials
- P_{nm} = Legendre functions of the first kind
- J_n, C_{nm}, S_{nm} = coefficients

Effect of J_2 Perturbation

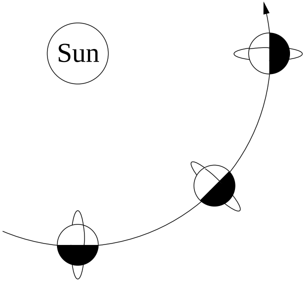
- J_2 is two orders of magnitude larger than others
  - Arises from the first-order deviation of the oblate earth from a sphere
- Small periodic changes in $a,\ e,\ i$ with

$$\dot{a} \simeq 0,\ \dot{e} \simeq 0,\ \dot{i} \simeq 0$$

- Secular changes in $\Omega,\ w,\ M$
  - Regression of nodes: $\dfrac{d\Omega}{dt} = -\dfrac{3}{2}\ n\ \dfrac{J_2 \cos i}{(1-e^2)^2} \left(\dfrac{R_e}{a}\right)^2$
  - Advance of perigee: $\dfrac{dw}{dt} = -\dfrac{3}{4}\ n\ J_2\ \dfrac{(1-5\cos^2 i)}{(1-e^2)^2} \left(\dfrac{R_e}{a}\right)^2$
  - Change in mean anomaly: $\dfrac{dM}{dt} = n + \dfrac{3nJ_2(3\cos^2 i - 1)}{4(1-e^2)^{3/2}} \left(\dfrac{R_e}{a}\right)^2$
- Superimposed periodic variations + secular and periodic variations due to higher order terms

Application: Sun Synchronous Orbits

- Orbits that have a nodal regression rate of $360°$ per year
- Orbital plane makes a constant angle with respect to sun



- Satellite revisits any point at the same local time
  - Useful for earth observation satellites
  - Solar illumination the same in pictures takes at different times

Launch to Rendezvous

- Launch a spacecraft to rendezvous with a space station already in the orbit
- Problem: Find time of launch so that both orbits are coplanar
  - Orbital plane changes after injection is expensive
  - Turning the launch vehicle into the required plane is also expensive
-
  -

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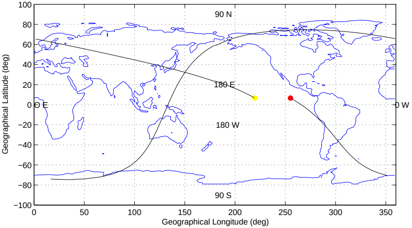
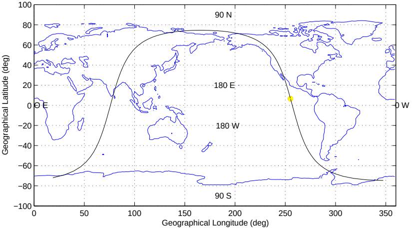
  - Orbital plane changes after injection is expensive
  - Turning the launch vehicle into the required plane is also expensive
- Solution: Launch when launch site lies in the space station orbital plane
  - For a given latitude, this occurs at most twice in every sidereal day

Projection of the Orbit

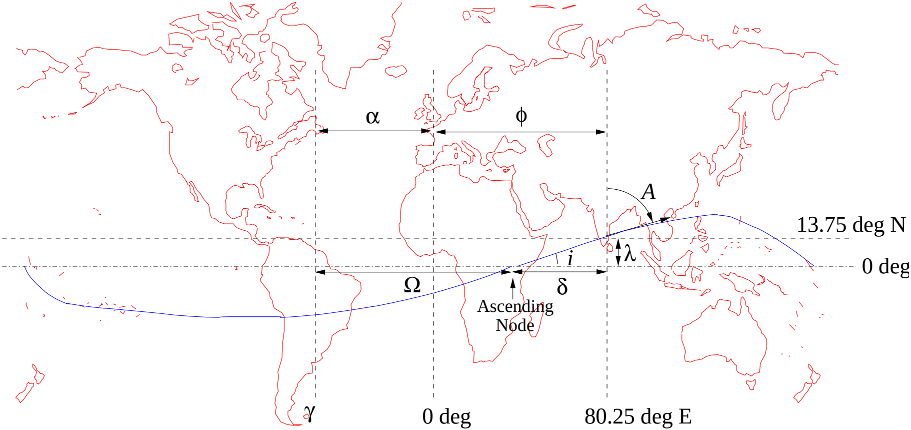


Ground Trace



Orbital Projection

Geometry of Coplanar Launch to Rendezvous



Launch Times for Rendezvous

$$\sin\delta = \tan\lambda\cot i$$

- Two solutions which add up to 180°
- Right ascension of launch site

$$= \alpha + \phi = \Omega + \delta$$

- Right ascension of Greenwich meridian

$$\alpha = \alpha_0 + \frac{2\pi}{T_{\text{sidereal}}}(t - t_0)$$

- Launch time

$$t = t_0 + \frac{T_{\text{sidereal}}}{2\pi}(\Omega + \delta - \phi - \alpha_0)$$

  - Two solutions
- Launch azimuth

$$\sin A = \frac{\cos i}{\cos\lambda}$$

Rotational Motion of Satellites

- Orbital dynamics: satellites treated as point masses
- Rotational motion as extended bodies has to be considered
- Attitude maneuvering
  - Pointing requirements of optical, communications, imaging payload
  - Solar panel orientation
  - Thruster orientation for orbital maneuvers and station keeping
- Treat satellite as a rigid body
  - Collection of particles such that the distance between any two remains fixed
  - Six degrees of freedom, 3 translational + 3 rotational

Translational Dynamics of a Rigid Body

- Consider a rigid arrangement of a finite number of particles
- For the i^{th} particle

$$\mathbf{F}_{i\ \text{ext}} + \sum_{j \neq i}^{N} \mathbf{F}_{ij} = m_i \mathbf{a}_i$$

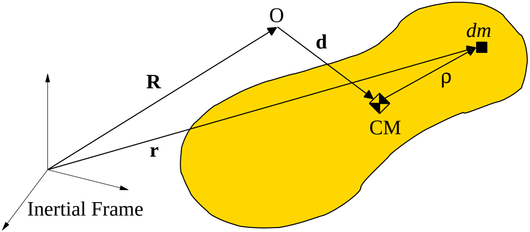
$$\mathbf{F}_{\text{ext}} \stackrel{\text{def}}{=} \sum_{i=1}^{N} \mathbf{F}_{i\ \text{ext}} + \underbrace{\sum_{i=1}^{N} \sum_{j \neq i}^{N} \mathbf{F}_{ij}}_{=0} = \sum_{i=1}^{N} m_i \mathbf{a}_i$$

$$\mathbf{F}_{\text{ext}} = M \mathbf{a}_{\text{cm}}, \ \mathbf{a}_{\text{cm}} = \frac{\sum m_i \mathbf{a}_i}{\sum m_i}$$

- Translates as a point particle of mass M located at center of mass under $\mathbf{F}_{\text{ext}}$
- Rotational and translational motions decoupled?

Rotational Dynamics of a Rigid Body

- Take moments of Newton's law about some convenient point



$$d\boldsymbol{F} = dm\, \ddot{\boldsymbol{r}} = dm\, (\ddot{\boldsymbol{R}} + \ddot{\boldsymbol{d}} + \ddot{\boldsymbol{\rho}})$$

$$d\boldsymbol{M}_O = (\boldsymbol{d} + \boldsymbol{\rho}) \times d\boldsymbol{F} = (\boldsymbol{d} + \boldsymbol{\rho}) \times (\ddot{\boldsymbol{R}} + \ddot{\boldsymbol{d}} + \ddot{\boldsymbol{\rho}})dm$$

Rotational Dynamics of a Rigid Body (cont'd)

$$\begin{aligned}
\boldsymbol{M}_O &= \int \left[(\boldsymbol{d}+\boldsymbol{\rho})\times(\ddot{\boldsymbol{d}}+\ddot{\boldsymbol{\rho}})\right] dm + \int (\boldsymbol{d}\times\ddot{\boldsymbol{R}})dm + \int (\boldsymbol{\rho}\times\ddot{\boldsymbol{R}})dm \\
&= \int \frac{d}{dt}\underbrace{\left[(\boldsymbol{d}+\boldsymbol{\rho})\times(\dot{\boldsymbol{d}}+\dot{\boldsymbol{\rho}})\right] dm}_{d\boldsymbol{H}_O} + (\boldsymbol{d}\times\ddot{\boldsymbol{R}})\underbrace{\int dm}_{\boldsymbol{M}} \\
&\qquad + \underbrace{\left(\int \boldsymbol{\rho}dm\right)}_{=\boldsymbol{0}}\times\ddot{\boldsymbol{R}} \\
&= \frac{d}{dt}\boldsymbol{H}_O + m(\boldsymbol{d}\times\ddot{\boldsymbol{R}})
\end{aligned}$$

- If O is inertially fixed $(\ddot{\mathbf{R}}=0)$ or the center of mass $(\mathbf{d}=0)$ then

  $$\boxed{\mathbf{M}_O = \frac{d}{dt}\mathbf{H}_O}$$ Attitude dynamics equation

  - $\mathbf{H}_O =$ moment about O of linear momentum relative to O

Attitude Representation

- Consider two right handed orthonormal frames
  - I with unit vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}$, B with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$
- Components of any vector $\mathbf{v}$ along I and B

$$(\mathbf{v})_{\mathrm{I}} = \begin{bmatrix}\mathbf{v.l} & \mathbf{v.m} & \mathbf{v.n}\end{bmatrix}^{\mathrm{T}}, \ (\mathbf{v})_{\mathrm{B}} = \begin{bmatrix}\mathbf{v.i} & \mathbf{v.j} & \mathbf{v.k}\end{bmatrix}^{\mathrm{T}}$$

$$\begin{aligned}(\mathbf{v})_{\mathrm{I}} &= \begin{bmatrix} \mathbf{v}_{\mathrm{B}}^1\ \mathbf{i.l} + \mathbf{v}_{\mathrm{B}}^2\ \mathbf{j.l} + \mathbf{v}_{\mathrm{B}}^3\ \mathbf{k.l} \\ \mathbf{v}_{\mathrm{B}}^1\ \mathbf{i.m} + \mathbf{v}_{\mathrm{B}}^2\ \mathbf{j.m} + \mathbf{v}_{\mathrm{B}}^3\ \mathbf{k.m} \\ \mathbf{v}_{\mathrm{B}}^1\ \mathbf{i.n} + \mathbf{v}_{\mathrm{B}}^2\ \mathbf{j.n} + \mathbf{v}_{\mathrm{B}}^3\ \mathbf{k.n} \end{bmatrix} = \begin{bmatrix} \mathbf{i.l} & \mathbf{j.l} & \mathbf{k.l} \\ \mathbf{i.m} & \mathbf{j.m} & \mathbf{k.m} \\ \mathbf{i.n} & \mathbf{j.n} & \mathbf{k.n} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\mathrm{B}}^1 \\ \mathbf{v}_{\mathrm{B}}^2 \\ \mathbf{v}_{\mathrm{B}}^3 \end{bmatrix} \\ &= \begin{bmatrix}(\mathbf{i})_{\mathrm{I}} & (\mathbf{j})_{\mathrm{I}} & (\mathbf{k})_{\mathrm{I}}\end{bmatrix} (\mathbf{v})_{\mathrm{B}}\end{aligned}$$

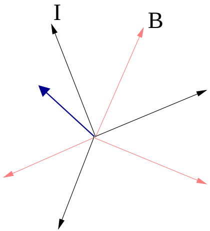
- There exists a unique matrix R such that $R(\mathbf{v})_{\mathrm{B}} = (\mathbf{v})_{\mathrm{I}}$ for every $\mathbf{v}$
- R determined solely by orientation of B relative to I
- R— special orthogonal matrix, rotation matrix, direction cosine matrix

$$R^{\mathrm{T}} R = I$$

$$\det R = 1$$

An Alternative Situation

- Rotate a frame to go from I to B
- What are the new I-components of a vector fixed to the moving frame?



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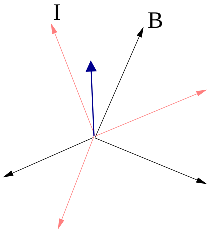

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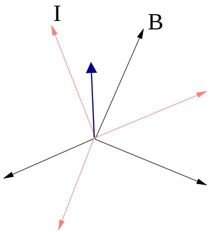


$$\begin{aligned}
\text{old I components} &= \text{new B components} \\
\text{new I components} &= R \cdot (\text{new B components}) \\
&= R \cdot (\text{old I components})
\end{aligned}$$

- R relates
  - Components of a given vector in two frames
  - Components of a rotated vector to its original components in the same frame

Rotation Matrix for an Elementary Rotation

- Rotate I about a unit vector $\mathbf{v}$ through an angle θ to obtain B

$$(\mathbf{v})_{\mathrm{I}} = (\mathbf{v})_{\mathrm{B}} \stackrel{\mathrm{def}}{=} v \in \mathbb{R}^3$$

- Transformation matrix

$$R(v,\theta) = I + (1-\cos\theta)(v\times)^2 + \sin\theta(v\times)$$

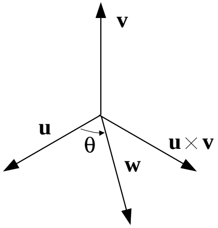
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (v\times) = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

  - If $(\mathbf{u})_{\mathrm{B}} = u \in \mathbb{R}^3$, then

$$(\mathbf{v}\times\mathbf{u})_{\mathrm{B}} = (v\times)u$$

Elementary Rotation (cont'd)

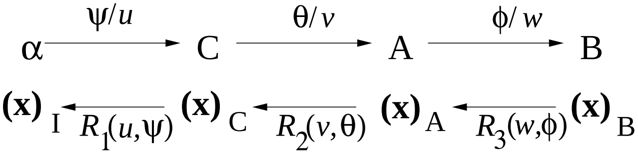
- Check:
  - $Rv = v$
  - Let $\mathbf{u} \perp \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u} = 1$. Let $\mathbf{w}$ be obtained from $\mathbf{u}$ by rotation about $\mathbf{v}$



$$(\mathbf{w})_{\mathrm{B}} = (\mathbf{u})_{\mathrm{I}} \overset{\mathrm{def}}{=} u$$

$$\begin{aligned}
\mathbf{w} &= \cos\theta\mathbf{u} + \sin\theta(\mathbf{v} \times \mathbf{u}) \\
&= \mathbf{u} + (1 - \cos\theta)\,(\mathbf{v} \times (\mathbf{v} \times \mathbf{u})) + \sin\theta(\mathbf{v} \times \mathbf{u}) \\
(\mathbf{w})_{\mathrm{I}} &= [I + (1 - \cos\theta)(v\times)^2 + \sin\theta(v\times)]u \\
&= R(\mathbf{w})_{\mathrm{B}}
\end{aligned}$$

Composite Rotations



$$R_{\text{composite}} = R_1(u,\psi)R_2(v,\theta)R_3(w,\phi)$$

- Composition of rotations $\sim$ matrix multiplication on the right
- Noncommutativity of matrix multiplication $\sim$ non commutativity of rotations

Relative Motion Between Frames

- Frame B rotates relative to I (not necessarily about a fixed axis)
- $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ vectors fixed in B, $\mathbf{r}_1, \mathbf{r}_2$ linearly independent
- $\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \dot{\mathbf{r}}_3$ instantaneous derivatives with respect to I
  - $\mathbf{r}_1 \cdot \mathbf{r}_1 = \text{constant} \Rightarrow \dot{\mathbf{r}}_1 \cdot \mathbf{r}_1 = 0$
  - $\mathbf{r}_1 \cdot \mathbf{r}_2 = \text{constant} \Rightarrow \dot{\mathbf{r}}_1 \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \dot{\mathbf{r}}_2 = 0$

$$\begin{aligned} \mathbf{r}_3 &= \alpha_1 \mathbf{r}_1 + \alpha_2 \mathbf{r}_2 + \alpha_3 (\mathbf{r}_1 \times \mathbf{r}_2) \\ \dot{\mathbf{r}}_3 &= \alpha_1 \dot{\mathbf{r}}_1 + \alpha_2 \dot{\mathbf{r}}_2 + \alpha_3 (\dot{\mathbf{r}}_1 \times \mathbf{r}_2 + \mathbf{r}_1 \times \dot{\mathbf{r}}_2) \end{aligned}$$

$$\implies (\dot{\mathbf{r}}_1 \times \dot{\mathbf{r}}_2) \cdot \dot{\mathbf{r}}_3 = 0$$

- Instantaneous derivative of every vector fixed in B lies in the plane perpendicular to $(\dot{\mathbf{r}}_1 \times \dot{\mathbf{r}}_2)$

$$\dot{\mathbf{r}} = \alpha(\mathbf{r})(\mathbf{e} \times \mathbf{r})$$

  - $\mathbf{e} =$ unit vector along $(\dot{\mathbf{r}}_1 \times \dot{\mathbf{r}}_2)$

Relative Motion (cont'd)

- $\mathbf{r}_1, \mathbf{r}_2$ vectors fixed in B, linearly independent from $\mathbf{e}$

$$\dot{\mathbf{r}}_1 = \alpha(\mathbf{r}_1)(\mathbf{e} \times \mathbf{r}_1)$$
$$\dot{\mathbf{r}}_2 = \alpha(\mathbf{r}_2)(\mathbf{e} \times \mathbf{r}_2)$$
$$\mathbf{r}_2 \cdot \dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2 \cdot \mathbf{r}_1 = 0 \Longrightarrow \alpha(\mathbf{r}_1) = \alpha(\mathbf{r}_2) = \text{constant}$$

- There exists a vector $\boldsymbol{\omega}$ such that instantaneous derivatives of vectors in B are given by

$$\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\boldsymbol{\omega} = \text{instantaneous angular velocity of } \mathrm{B} \text{ relative to } \mathrm{I}$$

Attitude Kinematics

- B rotates relative to I
- Instantaneous relative orientation described by rotation matrix $R(t)$
  - How does R vary?
- For any vector fixed in B,

$$(\dot{\mathbf{r}})_{\mathrm{I}} = \frac{d}{dt}(\mathbf{r})_{\mathrm{I}} = \frac{d}{dt}R(\mathbf{r})_{\mathrm{B}} = \dot{R}(\mathbf{r})_{\mathrm{B}} + R\underbrace{\frac{d}{dt}(\mathbf{r})_{\mathrm{B}}}_{=0}$$

$$(\dot{\boldsymbol{r}})_{\mathrm{I}} = (\boldsymbol{\omega} \times \boldsymbol{r})_{\mathrm{I}} = R(\boldsymbol{\omega} \times \boldsymbol{r})_{\mathrm{B}} = R(\omega\times)(\boldsymbol{r})_{\mathrm{B}}$$

$$\boxed{\dot{R} = R(\omega\times)}$$ Attitude kinematics equation

$$(\omega\times) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- ω =column vector of B components of instantaneous angular velocity of B relative to I

Attitude Kinematics Equation

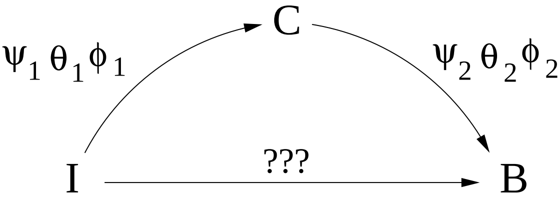
- Used for
  - Navigation
  - Control design
  - Simulation
- Solution involves integrating 9 differential equations
  - R contains 9 elements subject to 6 constraints, only 3 free parameters
- Question: Is it possible to parametrize rotation matrices with fewer parameters?
  - If yes, rewrite attitude kinematics in terms of fewer parameters

Euler Angles

- Fact: Given a sequence of unit vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ fixed in I such that no two consecutive vectors are linearly dependent, I can be rotated to any desired orientation by a sequence of three rotations, one each about $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$
  - For every rotation matrix R, there exist ψ, θ, ϕ such that

$$R = R_1(v_1, \psi)R_2(v_2, \theta)R_3(v_3, \phi)$$

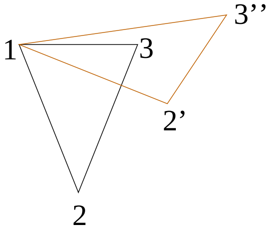
- Examples:
  - 3-2-1 Euler angles used in aircraft; $\mathbf{v}_1 = \mathbf{n}, \mathbf{v}_2 = \mathbf{m}, \mathbf{v}_3 = \mathbf{l}$
  - 3-1-3 Euler angles; $\mathbf{v}_1 = \mathbf{n}, \mathbf{v}_2 = \mathbf{l}, \mathbf{v}_3 = \mathbf{n}$
- Problem: Euler angles do not combine well for successive rotations



Axis-Angle Variables

- Euler's Theorem: Given two frames I and B, there exists an axis-angle pair $(\mathbf{v}, \theta)$ such that I coincides with B when rotated about $\mathbf{v}$ through θ
  - For every rotation matrix R, there exist $v \in \mathbb{R}^3, \theta \in [0, 2\pi)$ such that

$$R = R(v, \theta) = I + (1 - \cos\theta)(v\times)^2 + \sin\theta(v\times)$$



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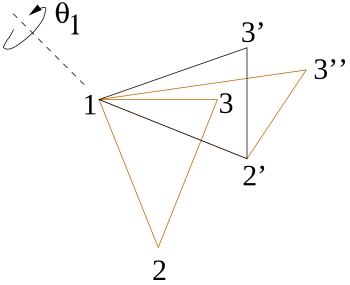

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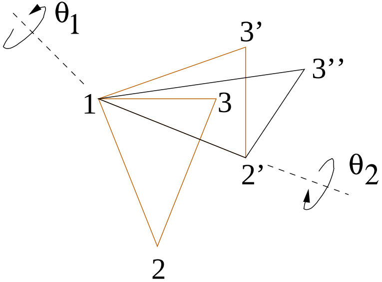

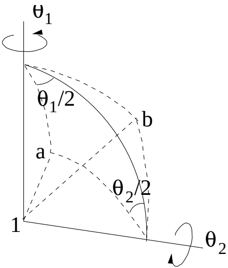

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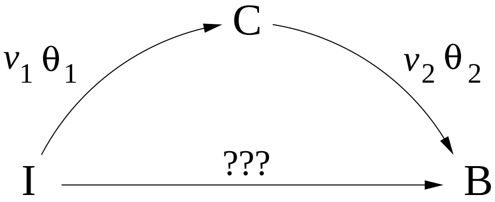
- Problem: Axis angle variables do not combine well for successive rotations



Leonhard Euler

Leonhard Euler
1707-1783

- Rigid body motion
- Fluid mechanics
- Solid mechanics
- Number theory
- Real and complex analysis
- Calculus of variations
- Differential geometry and topology
- Differential equations
- Mathematical notation

Attitude Kinematics with Euler Angle

- To obtain attitude kinematics in terms of ψ, θ, ϕ, substitute

$$R = R_1(v_1, \psi) R_2(v_2, \theta) R_3(v_3, \phi) \quad \text{in} \quad \dot{R} = R(w\times)$$

  - Solve for $\dot{\psi}, \dot{\theta}, \dot{\phi}$

- Example: 3-2-1 Euler angles

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}^{-1}}_{\text{singular at } \theta=\pm 90^\circ} \begin{bmatrix} \omega_3 \\ \omega_2 \\ \omega_1 \end{bmatrix}$$

- Fact: Every three-parameter representation of attitude possesses a kinematic singularity
  - Euler angles suitable only for simulating limited angular motion

Quaternions

$$q = \underbrace{q_0}_{\text{Real part}} + \underbrace{q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}}_{\text{Imaginary part}}$$

- To multiply quaternions, use

$$\mathrm{ii} = \mathrm{jj} = \mathrm{kk} = -1,\ \mathrm{ij} = -\mathrm{ji} = \mathrm{k},\ \mathrm{jk} = -\mathrm{kj} = \mathrm{i},\ \mathrm{ki} = -\mathrm{ik} = \mathrm{j}$$

  - Multiplication noncommutative
- Conjugate

$$\overline{q} = q_0 - q_1\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k}$$

- Magnitude $= \sqrt{q\overline{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$
- If $v = [v_1\ v_2\ v_3]^{\mathrm{T}}$, then define $\widehat{v} = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k}$

Quaternion Representation of Rotations

- If I is rotated through θ about unit vector v to obtain B, set

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\widehat{v}$$

  - Unit quaternion

- If $x_{\mathsf{B}} = (\mathbf{x})_{\mathsf{B}}$ and $x_{\mathsf{I}} = (\mathbf{x})_{\mathsf{I}}$ then

$$\widehat{x}_{\mathsf{I}} = q\widehat{x}_{\mathsf{B}}\overline{q}$$

  - No trigonometric formulae

$$\begin{aligned} q_{\mathsf{B}\to\mathsf{I}}^{-1} &= \overline{q}_{\mathsf{B}\to\mathsf{I}} = q_{\mathsf{I}\to\mathsf{B}} \\ q_{\mathsf{B}\to\mathsf{I}} &= q_{\mathsf{C}\to\mathsf{I}}q_{\mathsf{B}\to\mathsf{C}} \end{aligned}$$

William Rowan Hamilton

William Rowan Hamilton
1805-1865

- Algebra
- Optics
- Mechanics

Attitude Dynamics Revisited

$$\mathbf{M} = \dot{\mathbf{H}}$$

- - $\mathbf{M}$ = moment of external forces about center of mass
  - $\mathbf{H}$ = angular momentum of body about center of mass
  - Derivative with respect to inertial frame
- Let B be a body fixed frame with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$

$$\mathbf{H} = H_1\mathbf{i} + H_2\mathbf{j} + H_3\mathbf{k}, \quad H_\mathrm{B} = [H_1 \ H_2 \ H_3]^T$$

$$\boldsymbol{\omega} = \omega_1\widehat{i} + \omega_2\widehat{j} + \omega_3\widehat{k}, \quad \omega_\mathrm{B} = [\omega_1 \ \omega_2 \ \omega_3]^T$$

$$\mathbf{M} = M_1\widehat{i} + M_2\widehat{j} + M_3\widehat{k}, \quad M_\mathrm{B} = [M_1 \ M_2 \ M_3]^T$$

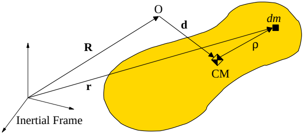
$$\dot{\mathbf{H}} = \dot{H}_1\widehat{i} + \dot{H}_2\widehat{j} + \dot{H}_3\widehat{k} + (\boldsymbol{\omega} \times \mathbf{H})$$

$$(\mathbf{M})_\mathrm{B} = (\dot{\mathbf{H}})_\mathrm{B} = \frac{d}{dt}H_\mathrm{B} \ + \ (\boldsymbol{\omega} \times \mathbf{H})_\mathrm{B} = \frac{d}{dt}H_\mathrm{B} \ + \ (\omega_\mathrm{B}\times)H_\mathrm{B}$$

$$\boxed{\frac{d}{dt}H_\mathrm{B} = -(\omega_\mathrm{B}\times)H_\mathrm{B} + M_\mathrm{B}}$$

- Attitude dynamics equation in terms of body components

Body Component of Angular Momentum



$$\int(\boldsymbol{\rho}\times\dot{\boldsymbol{\rho}})dm = \int\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})dm = -\int(\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\boldsymbol{\omega}))dm$$

Let

$$\boldsymbol{\rho}= x\hat{i} + y\hat{j} + z\hat{k},\ \ \rho_{\mathrm{B}} = [x\ y\ z\ ]^{\mathrm{T}},\ (\rho_{\mathsf{B}}\times) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

$$H_{\mathrm{B}} = -\int(\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\boldsymbol{\omega}))_{\mathrm{B}}dm = \int(\rho_{\mathrm{B}}\times)(\boldsymbol{\rho}\times\boldsymbol{\omega})_{\mathrm{B}}dm = -\int((\rho_{\mathrm{B}}\times)^2\omega_{\mathrm{B}})dm$$

$$I_{\mathrm{B}} = -\int(\rho_{\mathrm{B}}\times)^2dm = \begin{bmatrix} \int(y^2+z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int(x^2+z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int(x^2+y^2)dm \end{bmatrix}$$

$$\boxed{H_{\mathrm{B}} = I_{\mathrm{B}}\omega_{\mathrm{B}}}$$

- I_{B} = moment-of-inertia matrix about body axes

Principal Axes of Inertia

- Moment-of-inertia matrix about inertial axes

$$H_{\mathrm{I}} = RH_{\mathrm{B}} = RI_{\mathrm{B}}\omega_{\mathrm{B}} = RI_{\mathrm{B}}R^T\omega_{\mathrm{I}} = I_{\mathrm{I}} = RI_{\mathrm{B}}R^T$$

  - I_{I} varies as body rotates

- Two body frames, B amd B', related by rotation matix R

$$H_{\mathrm{B}} = I_{\mathrm{B}}\omega_{\mathrm{B}}, \qquad H_{\mathrm{B}'} = I_{\mathrm{B}'}\omega_{\mathrm{B}'}$$

$$H_{\mathrm{B}} = RH_{\mathrm{B}} = RI_{\mathrm{B}}\omega_{\mathrm{B}} = RI_{\mathrm{B}}R^T\omega_{\mathrm{B}'}, \qquad I_{\mathrm{B}\prime} = RI_{\mathrm{B}}R^T$$

- Fact: There exsists a rotation matrix R such that $RI_{\mathrm{B}}R^T = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$

- There exists a body frame B' such that $I_{\mathrm{B}'} =$ digonal
  - Corresponding axes are principal axes of inertia, I_1, I_2, I_3 are principal moment of inertia
  - Principal axes are along eigenvectors of I_{B}
  - Principal moments of inertia are eigenvalues of I_{B}

Euler's Equation for Rotational Dynamics

$$\frac{d}{dt}H_{\mathrm{B}} = -(\omega_{\mathrm{B}}\times)H_{\mathrm{B}} \; + \; M_{\mathrm{B}}$$

- Put $H_{\mathrm{B}} = I_{\mathrm{B}}\omega_{\mathrm{B}},\; I_{\mathrm{B}}$ constant

$$\boxed{I_{\mathrm{B}}\dot{\omega}_{\mathrm{B}} = -(\omega_{\mathrm{B}}\times)I_{\mathrm{B}}\omega_{\mathrm{B}} + M_{\mathrm{B}}}$$

  - Determine evolution of angular velocity component
  - Determine rotational motion together with attitude kinematics equation

- Euler's equation written for principal axes of inertia

$$\begin{aligned}
\dot{\omega}_1 &= -\frac{(I_3 - I_2)}{I_1}\omega_2\omega_3 + \frac{1}{I_1}M_1 \\
\dot{\omega}_2 &= -\frac{(I_1 - I_3)}{I_1}\omega_1\omega_3 + \frac{1}{I_2}M_2 \\
\dot{\omega}_3 &= -\frac{(I_2 - I_1)}{I_3}\omega_1\omega_2 + \frac{1}{I_3}M_3
\end{aligned}$$