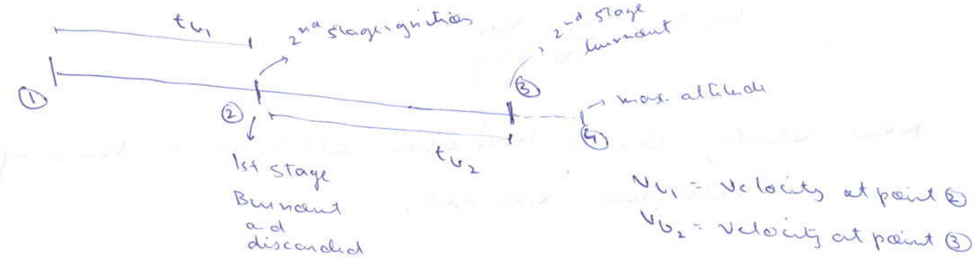
ANSHUL SHARMA
03d01001

Problem 4.4

Case I



t_{b_1} = Burnout time for 1st stage

t_{b_2} = Burnout time for 2^{nd} stage

For constant specific thrust under uniform gravity,

$$\dot{v} = -g - \frac{\dot{m} v_e}{m} \Rightarrow (v(t) - v_0) = -gt + v_e \ln \frac{m_i}{m(t)}$$

$$= -gt + v_e \ln e^{(\psi t/I_{sp})}$$

$$= -gt + v_e \frac{\psi t}{I_{sp}}$$

$$(v(t) - v_0) = -gt + \psi g t = (\psi - 1) g t \quad \text{①}$$

$\therefore \quad h_{12} = \frac{1}{2}(\psi - 1) g t_{b_1}^2 \quad$ (since $v_0 = 0$ for ① → ②)

Also, for rocket motion from ② → ③

$\cancel{v_{b_2} = v_{b_1}}$ $\quad h_{23} = \frac{1}{2}(\psi - 1) g t_{b_2}^2 + v_{b_1} t_{b_2}$

(since $v_0 = v_{b_1}$ for ② → ③)

Also, for rocket motion from ③ → ④

$$h_{34} = \frac{v_{b_2}^2}{2g}$$

$\therefore$ Culmination altitude

$$h_{c_1} = \frac{1}{2}(\psi - 1) g t_{b_1}^2 + \frac{1}{2}(\psi - 1) g t_{b_2}^2 + v_{b_1} t_{b_2} + \frac{v_{b_2}^2}{2g}$$

Now, $\dfrac{V_{v_2} - V_{v_1}}{t_{v_2}} = (\psi - 1)g$ — (using eq^n ①)

$$\therefore \quad t_{v_2} = \frac{V_{v_2} - V_{v_1}}{(\psi - 1)g}$$

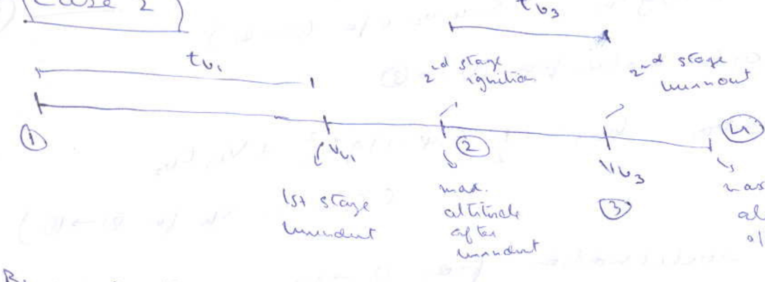
Now writing the culmination altitude in terms of velocities we get,

$$h_{c_1} = \underbrace{\frac{1}{2}(\psi-1)g\,t_{v_1}^2}_{\text{same for case (1) \& case (2)}} + \frac{1}{2}\cancel{(\psi-1)}g\,\frac{(V_{v_2} - V_{v_1})^2}{(\psi_0 - 1)^{\cancel{2}} g^2} + \frac{V_{v_1}(V_{v_2} - V_{v_1}) \times 2}{(\psi-1)g \times 2} + \frac{V_{v_2}^2}{2g}$$

$$\Rightarrow h_{c_1} = \frac{1}{2}(\psi-1)g\,t_{v_1}^2 + \frac{V_{v_2}^2 - 2\cancel{V_{v_1}V_{v_2}} + V_{v_1}^2 + 2\cancel{V_{v_1}V_{v_2}} - 2V_{v_1}^2}{2(\psi-1)g} + \frac{V_{v_2}^2}{2g}$$

$$\Rightarrow \boxed{h_{c_1} = \frac{1}{2}(\psi-1)g\,t_{v_1}^2 + \frac{V_{v_2}^2 - V_{v_1}^2}{2(\psi-1)g} + \frac{V_{v_2}^2}{2g}}$$

Case 2



V_{v_1} = burnout velocity after 1st stage

V_{v_3} = burnout velocity after 2nd stage

Burnout time for 1st stage in both cases will be same

Here $h_{12} = \frac{1}{2}(\psi-1)g\,t_{v_1}^2 + \frac{V_{v_1}^2}{2g}$

t_{v_1} = burnout time for 1st stage

t_{v_3} = burnout time for 2nd stage

At point ② $V = 0$, $\therefore$ the max. altitude after 1st stage burnout is $\frac{V_{v_1}^2}{2g}$

Also, $h_{24} = \frac{1}{2}(\psi-1)g t_{v_3}^2 + \frac{V_{v_3}^2}{2g}$ → altitude after 2nd stage burnout

∴ Culmination altitude $h_{c_2} =$

$$\frac{1}{2}(\psi-1)g t_{v_1}^2 + \frac{V_{v_1}^2}{2g} + \frac{1}{2}(\psi-1)g t_{v_3}^2 + \frac{V_{v_3}^2}{2g}$$

Now, $t_{v_3} = V_{v_3} / [(\psi-1)g]$

Writing h_{c_2} in terms of velocities we get,

$$h_{c_2} = \frac{1}{2}(\psi-1)g t_{v_1}^2 + \frac{V_{v_1}^2 + V_{v_3}^2}{2g} + \frac{1}{2}(\psi-1)g \frac{V_{v_3}^2}{(\psi-1)^2 g^2}$$

$$\Rightarrow \boxed{h_{c_2} = \frac{1}{2}(\psi-1)g t_{v_1}^2 + \frac{V_{v_1}^2 + V_{v_3}^2}{2g} + \frac{V_{v_3}^2}{2(\psi-1)g}}$$

Now, In case (1) we have

$$V_{v_2} = V_{v_1} + V_e \ln\left(\frac{m_{s_2} + m^* + m_{p_2}}{m_{s_2} + m^*}\right) \quad \text{—(2)}$$

In case (2) we have

$$V_{v_3} = 0 + \underbrace{V_e \ln\left(\frac{m_{s_2} + m^* + m_{p_2}}{m_{s_2} + m^*}\right)}_{V'} \quad \text{—(3)}$$

From (2) & (3) we have $V_{v_2} = V_{v_1} + V'$

$\therefore V_{v_3} = V'$ —(4)

Consider $h_{c_2} - h_{c_1}$, we get

$$h_{c_2} - h_{c_1} = \frac{V_{v_1}^2 + V_{v_3}^2 - V_{v_2}^2}{2g} + \frac{V_{v_3}^2 - V_{v_2}^2 + V_{v_1}^2}{2(\psi-1)g}$$

$$= \left(\frac{V_{v_1}^2 + V_{v_3}^2 - V_{v_2}^2}{2g}\right)\left[1 + \frac{1}{\psi-1}\right]$$

$$= \frac{\psi}{(\psi-1)2g}\left(v_{u_1}^2 + v_{u_3}^2 - v_{u_2}^2 \right)$$

Substituting $v_{u_3} = v'$ & $v_{u_2} = v_{u_1} + v'$ from (4)

we get

$$h_{c_2} - h_{c_1} = \frac{\psi}{(\psi-1)2g}\left(v_{u_1}^2 + v'^2 - \left(v_{u_1}^2 + v'^2 + 2v_{u_1}v' \right) \right)$$

$$\Rightarrow h_{c_2} - h_{c_1} = \frac{\psi}{(\psi-1)2g}\left[-2v_{u_1}v' \right]$$

$$\Rightarrow h_{c_2} - h_{c_1} = -\left[\frac{\psi \cancel{2} v_{u_1} v'}{(\psi-1)\cancel{2}g} \right] = -\frac{\psi v_{u_1} v'}{(\psi-1)g}$$

$\therefore \quad h_{c_2} - h_{c_1} < 0 \quad \Rightarrow h_{c_2} < h_{c_1}$

$\Rightarrow$ Culmination altitude will be higher for the 1st trajectory.

Harsha . V
04001014

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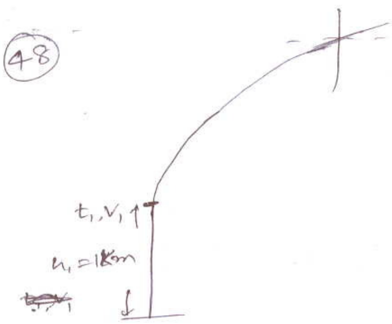


$\psi = 2.5$

$I_{sp} = 300 s$

$\Gamma_b = 55^\circ$

$\Gamma_1 = 90 - 5^\circ = 85^\circ.$

As we know for vertical ascent

$$h_1 = \frac{1}{2}(\psi - 1) g t_1^2 = 1 \text{ km}$$

$$\Rightarrow t_1 = 11.66 \text{ s}$$

$$v_1 = (\psi - 1) g t_1$$

$$= 171.46 \text{ m/s}.$$

In the 2^{nd} phase of trajectory gravity turn is initiated. making $\Gamma_1 = 90^\circ - 5^\circ$

$$= 85^\circ.$$

$$\Gamma_b = 55^\circ$$

$$v_1 = 171.46 \text{ m/s}$$

$$t_1 = 11.66 s$$

using the relation

$$V(t) \cos \Gamma(t) \left[\frac{1 + \sin \Gamma(t)}{1 - \sin \Gamma(t)}\right]^{\psi/2} = v_\phi \cos \Gamma_\phi \left[\frac{1 + \sin \Gamma_\phi}{1 - \sin \Gamma_\phi}\right]^{\psi/2}$$

$$\Rightarrow v_b \cos \underset{\substack{\downarrow \\ 55}}{\Gamma_b} \times \left[\frac{1 + \sin 55}{1 - \sin 55}\right]^{2.5/2} = 171.46 \times \left[\frac{1 + \sin 85}{1 - \sin 85}\right]^{2.5/2} \times \cos 85$$

$$\Rightarrow \quad v_b = 3651.2 \text{ m/s}$$

Burnout time is given by

$$t_b - t_1 = \frac{1}{g(\psi^2-1)}\left[v_b(\psi + \sin\Gamma_b) - v_1(\psi_1 + \sin\Gamma_1)\right]$$

$$t_b - 11.66 = 223.89$$

$$\Rightarrow \quad t_b = 235.6 \text{ s}$$

$$z_b - z_0 = \frac{1}{4g(\psi^2-1)}\left\{\begin{aligned} &v_b^2\left[1+\sin^2\Gamma_b + 2\psi\sin\Gamma_b\right] \\ &-v_1^2\left[1+\sin^2\Gamma_1 + 2\psi\sin\Gamma_1\right]\end{aligned}\right\}$$

$$z_b - 1000 = 372563$$

$$\Rightarrow z_b = 373563 \text{ m} = 374 \text{ km}$$

$$x_b - x_0 = \frac{1}{g(4\psi^2-1)}\left[\begin{aligned} &v_b^2\cos\Gamma_b(2\psi + \sin\Gamma_b) \\ &\quad - v_0^2\cos\Gamma_0(2\psi + \sin\Gamma_0)\end{aligned}\right]$$

$$= 189118.9 \text{ m}$$

$$\Rightarrow x_b = 189. \text{ km}$$

$$m(t) = m_i e^{-\gamma t/I_{sp}}$$

$$m_f = m_i e^{-\gamma t_b/I_{sp}}$$

$$= m_i \times 0.14$$

$$m_p = m_i - m_f$$

$$= \frac{m_f}{0.14} - m_f \quad = m_f\left(\frac{1}{0.14} - 1\right)$$

$$= m_f \times 6.14$$

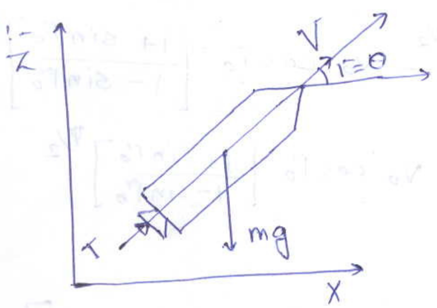
$$= 3000 \times 6.14$$

$$= 18,420 \text{ kgs}$$

$\Rightarrow$ mass of propellant $= \underline{18,420 \text{ kgs}}$

Q. 50) A rocket ascends in uniform gravitational field ($g = 9.8\ m/s^2$) ~~Given~~ in vaccum at a constant specific thrust 2.5 and a constant specific impulse of 300 s. The rocket is to be prepared for a different mission. The new mission involves ascending vertically to an altitude of 1 km, where a gravity turn is initiated by turning the velocity vector by an angle of 5°, what mass of propellant should the rocket carry so that it still achieves the same burnout flight path angle (i.e. 55°)? calculate the burnout speed, altitude and downrange, and the total burn time for the new mission.

soln:-



where
V: speed
Γ: flight path angle
z: altitude
X: Downrange
Assumption:- T = 0 (AoA = 0)

$\dot{V} = T/m - g \sin \Gamma$

$\dot{\Gamma} = \frac{-g}{V} \cos \Gamma$

$\dot{x} = V \cos \Gamma$

$\dot{z} = V \sin \Gamma$

Given:

$g = 9.8\ m/s^2$

$\psi = 2.5$

$I_{sp} = 300\ s.$

$\Gamma_0 = \pi/2 - 5° = 85°$

$\Gamma_b = 55°$

Vertical ascend to an altitude of 1 km; ($\Gamma = \pi/2$)

$\dot{z} = V \sin \pi/2 = V$

$\dot{V} = T/m - g \sin \pi/2 = \psi g - g = (\psi - 1) g = 14.7\ m/s^2$

integrating above eqn (assuming $V(0) = 0$)

$\Rightarrow V(t) = (\psi - 1) g t$

$\Rightarrow \dot{z}(t) = (\psi - 1) g t$

now, integrating above equation, we get

$\Rightarrow z(t_1) - \cancel{z(0)}^{0} = \frac{(\psi - 1)}{2} g t_1^2$

$(z(t_1) = 1\ km)$

P. T. O.

$\Rightarrow t_1 = \sqrt{\dfrac{2 \cdot z(t_1)}{(\psi-1)\, g}} = \cancel{0.3689}\ 11.66\ s.$

$v(t_1)$

$\Rightarrow \cancel{v} = (\psi - 1)\, g\, t_1 = 171.402\ m/s$

Now, gravity turn is initiated by turning the velocity vector by an angle of $5°$.

$\Gamma_0 = 90° - 5° = 85°$

$\Gamma_b = 55°$

$V_0 = V(t_1) = 171.402\ m/s$

$t_0 = t_1 = 11.66\ s$

(when $\Gamma < \pi/2 \Rightarrow \dot{\Gamma} < 0$, we have to bring Γ down ~~upto~~ to $55°$ hence $\Gamma_0 = 85°$ (not $95°$))

Using

$$V(t) \cos\Gamma(t) \left[\frac{1+\sin\Gamma(t)}{1-\sin\Gamma(t)}\right]^{\psi/2} = V_0 \cos\Gamma_0 \left[\frac{1+\sin\Gamma_0}{1-\sin\Gamma_0}\right]^{\psi/2}$$

$$\Rightarrow V_b \cos\Gamma_b \left[\frac{1+\sin\Gamma_b}{1-\sin\Gamma_b}\right]^{\psi/2} = V_0 \cos\Gamma_0 \left[\frac{1+\sin\Gamma_0}{1-\sin\Gamma_0}\right]^{\psi/2}$$

$\Rightarrow V_b = 3650.005\ m/s$

Now, using

$$\Rightarrow t_b - t_0 = \frac{1}{g(\psi^2-1)}\left[V_b(\psi + \sin\Gamma_b) - V_0(\psi + \sin\Gamma_0)\right]$$

$\Rightarrow t_b = 223.82\ s$

$$\Rightarrow Z_b = Z_0 + \frac{1}{4g(\psi^2-1)}\left\{V_b^2\left[1+\sin^2\Gamma_b + 2\psi\sin\Gamma_b\right] - V_0^2\left[1+\sin^2\Gamma_0 + 2\psi\sin\Gamma_0\right]\right\}$$

$= 1000 + 429623.058 = 430.62\ km$

$$\Rightarrow X_b = X_0 + \frac{1}{g(4\psi^2-1)}\left[V_b^2\cos\Gamma_b(2\psi+\sin\Gamma_b) - V_0^2\cos\Gamma_0(2\psi+\sin\Gamma_0)\right]$$

$= X_0 + 188{,}995.1373\ m$

($\dot{X}$ during vertical ascend was zero, i.e. X_0 $X(0) = 0$)

$\Rightarrow X_b = 188.99\ km$

$\Rightarrow m(t) = m_i \, e^{-\psi' t / I_{sp}}$

$\Rightarrow m_f = m_i \, e^{-\psi t_b / I_{sp}} = m_i (0.15487)$

$\Rightarrow m_p = m_i - m_f = m_i (1 - 0.15487)$

$= m_f \dfrac{(1-0.15487)}{(0.15487)}$

$m_f = 20{,}000 \text{ kg} - 18000 \text{ kg} = 2000 \text{ kg}$
(from previous problem)

hence,
mass of preopellant required is
$m_p = 10{,}875.31 \text{ kg}$

Mass of propellant required $(m_p) \simeq 10{,}875.3 \text{ kg}$
burnout speed $(V_b) = 3650.005 \text{ m/s} \simeq 13140 \text{ km/h}$
altitude $(Z_b) = 430.62 \text{ km}$
down range $(X_b) = 188.99 \text{ km}$
total burn time $(t_b) = 223.82$ seconds

submitted by :-
Kirtan S. Minpwal (04001001)

Pr. 51

(AE 415)

Rohan Kothari
04001002

$$B_1 = R_x B$$

$R_x \rightarrow$ Rotation matrix about x axis of B

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & \sin 45 & \cos 45 \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \text{ — Ans.}$$

$$B_2 = R_v B_1$$

$$V = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$; $R_v \rightarrow$ Rotation matrix about v in B_1.

$$R_v = \begin{bmatrix} 0 & -V_z & V_Y \\ V_z & 0 & -V_x \\ -V_Y & V_x & 0 \end{bmatrix} \sin\theta + (I - VV^T)\cos\theta + VV^T$$

$$= \begin{bmatrix} 0 & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 \end{bmatrix} \sin 30 + \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \right) \cos 30$$

$$+ \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/2\sqrt{3} & 1/2\sqrt{3} \\ 1/2\sqrt{3} & 0 & -1/2\sqrt{3} \\ -1/2\sqrt{3} & 1/2\sqrt{3} & 0 \end{bmatrix} + \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \frac{\sqrt{3}}{2} + \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sqrt{3}/6 & \sqrt{3}/6 \\ \sqrt{3}/6 & 0 & -\sqrt{3}/6 \\ -\sqrt{3}/6 & \sqrt{3}/6 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{3}}{3} & -\sqrt{3}/6 & -\sqrt{3}/6 \\ -\sqrt{3}/6 & \sqrt{3}/3 & -\sqrt{3}/6 \\ -\sqrt{3}/6 & -\sqrt{3}/6 & \sqrt{3}/3 \end{bmatrix} + \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} + \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$R_v = \begin{bmatrix} \frac{\sqrt{3}+1}{3} & \frac{1-\sqrt{3}}{3} & 1/3 \\ 1/3 & \frac{\sqrt{3}+1}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & 1/3 & \frac{\sqrt{3}+1}{3} \end{bmatrix} \text{— Ans}$$

$$B_2 = R_{B_2B}\, B$$

$$R_{B_2B} = R_v R_x.$$

$$= \begin{bmatrix} \frac{\sqrt{3}+1}{3} & \frac{1-\sqrt{3}}{3} & 1/3 \\ 1/3 & \frac{\sqrt{3}+1}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & 1/3 & \frac{\sqrt{3}+1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}+1}{3} & \frac{2-\sqrt{3}}{3\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1/3 & \sqrt{2/3} & -\sqrt{2/3} \\ \frac{1-\sqrt{3}}{3} & \frac{\sqrt{3}+2}{3\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \text{— Ans.}$$

AMIT KUMAR GARG 04001003

Q 52. Given :-

$q = 0.5 \, (1 + \hat{i} + \hat{j} + \hat{k})$

$\because \quad q = \cos \theta/_2 + v_1 \sin \theta/_2 \, \hat{i} + v_2 \sin \theta/_2 \, \hat{j} + v_3 \sin \theta/_2 \, \hat{k}$

where $v = [v_1 \ v_2 \ v_3]^T$

'V is the vector having components v_1, v_2, v_3 in B and B is obtained from I by a rotation of angle θ about a unit vector v.

$\Rightarrow \cos\theta/_2 = 0.5 \quad \Rightarrow \theta/_2 = 60^\circ \quad \Rightarrow \theta = 120^\circ$

$v_1 \sin\theta/_2 = 1/_2 \quad \Rightarrow \quad v_1 = v_2 = v_3 = 1/\sqrt{3}$

Using problem 58:-

$R = I + (1-\cos\theta)(v_\times)^2 + \sin\theta \, (v_\times)$

where

$$v_\times = \begin{bmatrix} 0 & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 \end{bmatrix} \Rightarrow (v_\times)^2 = \frac{1}{3}\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$1-\cos\theta = 1 + 1/_2 = 3/_2 \qquad \sin\theta = \sqrt{3}/_2$

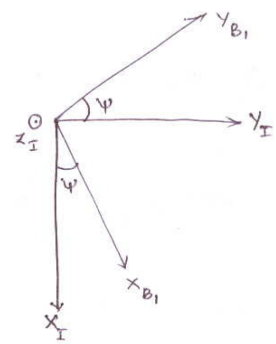
$$\Rightarrow R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1/_2 & 1/_2 \\ 1/_2 & -1 & 1/_2 \\ 1/_2 & 1/_2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1/_2 & 1/_2 \\ 1/_2 & 0 & -1/_2 \\ -1/_2 & 1/_2 & 0 \end{bmatrix}$$

$$\boxed{R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \text{Rotation matrix}}$$

15th Apr., '08. AE 415 - Tutorial

Submitted by:
Nirmala Uppu
04001012

Pb 55) $I \xrightarrow{R_1} B_1 \xrightarrow{R_2} B_2 \xrightarrow{R_3} B$



$$(\bar{V})_I = \bar{V}_{x_I} + \bar{V}_{y_I} + \bar{V}_{z_I}$$

$$(\bar{V})_{B_1} = \bar{V}_{x_{B_1}} + \bar{V}_{y_{B_1}} + \bar{V}_{z_{B_1}}$$

$$(\bar{V})_{B_2} = \bar{V}_{x_{B_2}} + \bar{V}_{y_{B_2}} + \bar{V}_{z_{B_2}}$$

$$(\bar{V})_{B} = \bar{V}_{x_{B}} + \bar{V}_{y_{B}} + \bar{V}_{z_{B}}$$

R_1: $\bar{V}_{x_{B_1}} = \cos\psi \times \bar{V}_{x_I} + \sin\psi \times \bar{V}_{y_I} + 0 \times \bar{V}_{z_I}$

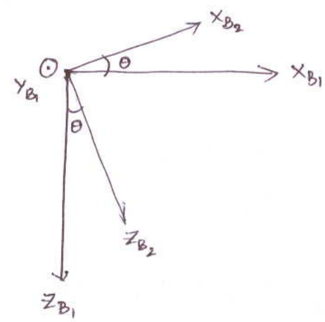
$\bar{V}_{y_{B_1}} = -\sin\psi \times \bar{V}_{x_I} + \cos\psi \times \bar{V}_{y_I} + 0 \times \bar{V}_{z_I}$

$\bar{V}_{z_{B_1}} = 0 \times \bar{V}_{x_I} + 0 \times \bar{V}_{y_I} + 1 \times \bar{V}_{z_I}$

$$\Rightarrow R_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R_2 :

$$\bar{V}_{x_{B_2}} = \cos\theta \times \bar{V}_{x_{B_1}} + 0 \times \bar{V}_{y_{B_1}} - \sin\theta \times \bar{V}_{z_{B_1}}$$

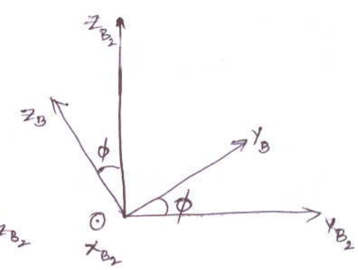


$$\bar{V}_{y_{B_2}} = 0 \times \bar{V}_{x_{B_1}} + 1 \times \bar{V}_{y_{B_1}} + 0 \times \bar{V}_{z_{B_1}}$$

$$\bar{V}_{z_{B_2}} = \sin\theta \times \bar{V}_{x_{B_1}} + 0 \times \bar{V}_{y_{B_1}} + \cos\theta \times \bar{V}_{z_{B_1}}$$

$$\Rightarrow R_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

R_3 :



$$\bar{V}_{x_B} = 1 \times \bar{V}_{x_{B_2}} + 0 \times \bar{V}_{y_{B_2}} + 0 \times \bar{V}_{z_{B_2}}$$

$$\bar{V}_{y_B} = 0 \times \bar{V}_{x_{B_2}} + \cos\phi \times \bar{V}_{y_{B_2}} + \sin\phi \times \bar{V}_{z_{B_2}}$$

$$\bar{V}_{z_B} = 0 \times \bar{V}_{x_{B_2}} + (-\sin\phi) \times \bar{V}_{y_{B_2}} + \cos\phi \times \bar{V}_{z_{B_2}}$$

$$\Rightarrow R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$\therefore$ R $\quad \overline{V}_{B_1} = R_1 \overline{V}_I$

$$\overline{V}_{B_2} = R_2 \overline{V}_{B_1}$$

$$\overline{V}_B = R_3 \overline{V}_{B_2}$$

$$\Rightarrow \overline{V}_B = (R_3 \cdot R_2 \cdot R_1) \overline{V}_I$$

$$\therefore R = R_3 \cdot R_2 \cdot R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\phi & S_\phi \\ 0 & -S_\phi & C_\phi \end{bmatrix} \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\phi & S_\phi \\ 0 & -S_\phi & C_\phi \end{bmatrix} \begin{bmatrix} C_\theta C_\psi & C_\theta S_\psi & -S_\theta \\ -S_\psi & C_\psi & 0 \\ S_\theta C_\psi & S_\theta S_\psi & C_\theta \end{bmatrix}$$

$$= \begin{bmatrix} C_\theta C_\psi & C_\theta S_\psi & -S_\theta \\ -C_\phi S_\psi + S_\phi S_\theta C_\psi & C_\phi C_\psi + S_\phi S_\theta S_\psi & S_\phi C_\theta \\ S_\phi S_\psi + C_\phi S_\theta C_\psi & -S_\phi C_\psi + C_\phi S_\theta S_\psi & C_\phi C_\theta \end{bmatrix}$$

AE 415 Tute Assignment

Rahul Goel
04001009

Q 59 Given vectors a & ω

Let thr components in B-frame be:

$$a = [a_1 \ a_2 \ a_3]$$

$$\omega = [\omega_1 \ \omega_2 \ \omega_3]$$

N.B. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors in I and $\hat{l}, \hat{m}, \hat{n}$ be unit vectors in B.

Now consider $\dot{\omega} = a \times \omega$

Part 1

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\dot{\omega}_1 = -a_3\omega_2 + a_2\omega_3 \quad -(1)$$

$$\dot{\omega}_2 = a_3\omega_1 - a_1\omega_2 \quad -(2)$$

$$\dot{\omega}_3 = -a_2\omega_1 + a_1\omega_2 \quad -(3)$$

Multiply eqns. (1), (2), (3) by $\omega_1, \omega_2, \omega_3$ respectively & add

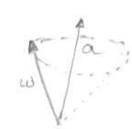
$$\Rightarrow \quad \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 + \omega_3\dot{\omega}_3 = 0$$

$$\Rightarrow \quad \omega_1^2 + \omega_2^2 + \omega_3^2 = \text{Constant}$$

This implies that the length of vector 'ω' is not changing in B-frame. And the motion of 'ω' is governed by eqns. 1-3.

Now we can see that $\dot{\omega}$ vector will be ⊥ to both a & ω. Now vector a is fixed, so basically vector ω will be rotating about vector a. And the tip of vector ω will trace a circle about a.

Vector ω will trace out a surface of a cone about vector a in the frame B.



Part 2

In this we need to show that the components of vector $a+\omega$ in the frame I remain constant.

So consider :

$$(a+\omega)_I = R\,(a+\omega)_B$$
$$= R\,(a+\omega)$$

Take derivative on both sides.

$$\dot{(a+\omega)}_I = \dot{R}\,(a+\omega) + R\,(\dot{a}+\dot{\omega})$$

$$= R\omega\times(a+\omega) + R\dot{\omega}$$

$$= R\,\omega\times a + R\,a\times\omega$$

$$= R\left(\omega\times a + a\times\omega\right)$$

using
$\dot{a} = 0$
$\dot{R} = R\Omega$
$= R\omega\times$
$\omega\times\omega = 0$
$\dot{\omega} = a\times\omega$

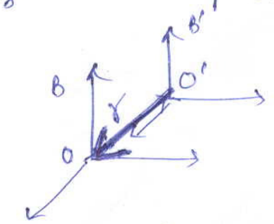
$$\dot{(a+\omega)}_I = 0$$

Hence $(a+\omega)_I$ is a constant.

17/04/08 Space Flight Mechanics Tannishtha

04d01010

Q.7]



$OO' = \gamma \in \mathbb{R}^3$ (given)

$\gamma \equiv [\gamma_1 \; \gamma_2 \; \gamma_3]$

B and B' are parallel frames

Now we know that,

$$I_{B'} = \begin{bmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (y^2+x^2)\,dm \end{bmatrix}$$

Similar term for I_B' in terms x', y', z'

In B' and B,

$$\left.\begin{aligned} x' &= x+\gamma_1 \\ y' &= y+\gamma_2 \\ z' &= z+\gamma_3 \end{aligned}\right\} (I)$$

Substituting (I) in I_B' we get,

$$I_B' = I_B + \underbrace{\begin{bmatrix} \int(\gamma_2^2+\gamma_3^2)\,dm & -\int \gamma_1\gamma_2\,dm & -\int \gamma_1\gamma_3\,dm \\ -\int \gamma_1\gamma_2\,dm & \int(\gamma_1^2+\gamma_3^2)\,dm & -\int \gamma_2\gamma_3\,dm \\ -\int \gamma_1\gamma_3\,dm & -\int \gamma_2\gamma_3\,dm & \int(\gamma_2^2+\gamma_1^2)\,dm \end{bmatrix}}_{I_\gamma}$$

$$+ \begin{bmatrix} 2\int\gamma_2 y\,dm + 2\int\gamma_3 z\,dm & -\left(\int\gamma_2 x\,dm + \int\gamma_1 y\,dm\right) & -\left(\int\gamma_1 z\,dm + \int\gamma_3 x\,dm\right) \\ -\int x\gamma_2\,dm - \int y\gamma_1\,dm & 2\int\gamma_1 x\,dm + 2\int\gamma_3 z\,dm & -\int z\gamma_2\,dm - \int\gamma_3 y\,dm \\ -\int x\gamma_3\,dm - \int z\gamma_1\,dm & -\int y\gamma_3\,dm - \int z\gamma_2\,dm & 2\int\gamma_2 y\,dm + 2\int\gamma_1 x\,dm \end{bmatrix}$$

☆ (If we assume CoM to be at O then $\int y\,dm = 0$; $\int x\,dm = 0$; $\int z\,dm = 0$)

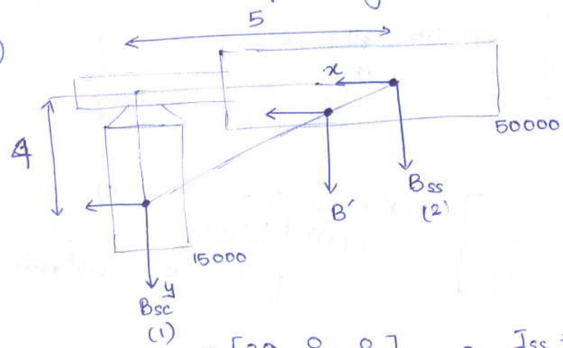
$\Rightarrow I_B' = I_B + I_\gamma$

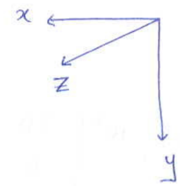
$$\therefore I_B' = I_B + I_\gamma + \int y\,dm \begin{bmatrix} 2\gamma_2 & -\gamma_1 & 0 \\ -\gamma_1 & 0 & -\gamma_3 \\ 0 & -\gamma_3 & 2\gamma_2 \end{bmatrix} + \int x\,dm \begin{bmatrix} 0 & -\gamma_2 & -\gamma_3 \\ -\gamma_2 & 2\gamma_1 & 0 \\ -\gamma_3 & 0 & 2\gamma_1 \end{bmatrix} + \int z\,dm \begin{bmatrix} 2\gamma_3 & 0 & -\gamma_1 \\ 0 & 2\gamma_3 & -\gamma_2 \\ -\gamma_1 & -\gamma_2 & 0 \end{bmatrix}$$

☆

AE - 415 Space Flight Mechanics. Dahlia N.

62





$$I_{sc} = 10^3 \begin{bmatrix} 20 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 20 \end{bmatrix} kg\,m^2 \qquad I_{ss} = 10^3 \begin{bmatrix} 90 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} kg\,m^2$$

x, z similar
axis $\equiv y$

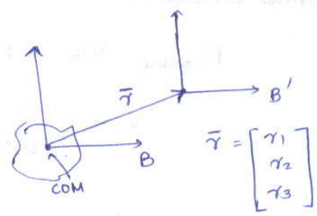
y, z similar.
axis $\equiv x$

From G1, $$I_{B'} = I_B + m \begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix}$$

Now, from Bss,

$$COM_{Bss\,frame} = \begin{bmatrix} \frac{15(5) + 50(0)}{65} \\ \frac{15(4) + 50(0)}{65} \\ 0 \end{bmatrix}$$

if



$x' = -r_1 + x$
$y' = -r_2 + y$
$z' = -r_3 + z$

$$\therefore \bar{r}_{B', B_{ss}} = \frac{1}{13} \begin{bmatrix} 15 \\ 12 \\ 0 \end{bmatrix}$$

Similarly, $$\bar{r}_{B', B_{sc}} = \frac{1}{13} \begin{bmatrix} -50 \\ -40 \\ 0 \end{bmatrix}, \quad \bar{r}_{B_{sc}, B'} = \frac{1}{13} \begin{bmatrix} 50 \\ 40 \\ 0 \end{bmatrix}$$

$$\therefore I_{SC,B'} = 10^3 \begin{bmatrix} 20 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 20 \end{bmatrix} + \frac{15}{169} \times 10^3 \begin{bmatrix} (40)^2 & -2000 & 0 \\ -2000 & (50)^2 & 0 \\ 0 & 0 & (40)^2 + (50)^2 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} \cancel{369.231} & \cancel{-461.538} & \cancel{0} \\ \cancel{-461.538} & & \cancel{0} \\ \cancel{0} & & 0 \end{bmatrix}$$

$$\therefore I_{sc,B'} = 10^3 \begin{bmatrix} 162.012 & -177.515 & 0 \\ -177.515 & 233.893 & 0 \\ 0 & 0 & 383.905 \end{bmatrix}$$

$$I_{ss,B'} = 10^3 \begin{bmatrix} 90 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} + 10^3 \times \frac{50}{(13)^2} \begin{bmatrix} (12)^2 & -(12)(15) & 0 \\ -(12)(15) & (15)^2 & 0 \\ 0 & 0 & (12)^2+(15)^2 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 132.603 & -53.254 & 0 \\ -53.254 & 116.568 & 0 \\ 0 & 0 & 159.172 \end{bmatrix}$$

I_{docked}

$$\therefore \boxed{I_{docked\ conf}} = I_{sc,B'} + I_{ss\,B'} = \boxed{\begin{bmatrix} 294.615 & -230.769 & 0 \\ -230.769 & 350.461 & 0 \\ 0 & 0 & 543.077 \end{bmatrix} \times 10^3 \ kgm^2}$$

✓

Linear momentum conserved. In inertial frame,

$$M_{docked} \cdot V_{CM} = M_{sc} \cdot V_{sc}$$

V_{com}

$$V_{CM} = \frac{15}{65} \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix} m/s \text{ in intertial frame} = \boxed{\begin{bmatrix} 0 \\ -2.3077 \\ 0 \end{bmatrix} m/s}$$

Angular momentum conserved. $H_i = \frac{1}{13} \begin{bmatrix} 0 & 0 & +40 \\ 0 & 0 & -50 \\ -40 & +50 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix} \times 15000$

$$\begin{bmatrix} 0 \\ 0 \\ -576923.1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -(15000)(10)\frac{(50)}{13} \end{bmatrix} = I_{docked} \cdot \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 271.184 & -170.769 & 0 \\ -170.769 & 350.461 & 0 \\ 0 & 0 & 543.077 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$\rightarrow$ ~~Hdocked~~ $\boxed{H_{com} = I_{docked} \cdot \overline{\omega}}$

Angular momentum about COM.

Solving for simultaneous equations, (or using matlab)

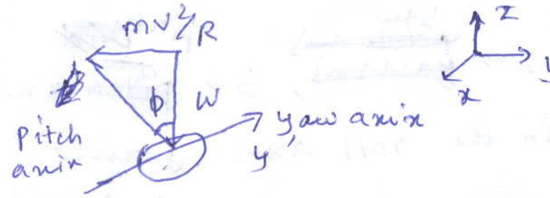
we get $\overline{\omega}$ = angular vel of docked conf. just after docking =

H_{docked} ω_{com}

$$\therefore \boxed{H_{com} = \begin{bmatrix} 0 \\ 0 \\ -576923.1 \end{bmatrix} \frac{kgm^2}{s}} \Rightarrow \boxed{\overline{\omega}_{com} = [J_{docked}]^{-1} \cdot H_{com} = \begin{bmatrix} 0 \\ 0 \\ -1.0623 \end{bmatrix} s^{-1}}$$

AE 415 - Spaceflight Mechanics

63)



Inertial 04d01007

Chetan Lalwani

roll axis out of the plane

$$\therefore \quad \tan\phi = \frac{W}{g}\,\frac{V^2}{RW} = \frac{V^2}{Rg} \quad \text{for coordinated turn}$$

The ~~banked aircraft makes angle~~ $\cancel{\phi}$ ~~with~~

The aircraft is rotated by an angle ϕ about the x axis (roll axis). $\therefore$ ~~The body frame~~ ~~roll axis makes an angle~~ $\cancel{\phi}$ ~~with the inertial x axis.~~

$$\therefore \quad \vec{\omega}_I = R(\phi_{\text{about } x})\ \vec{\omega}_B$$

$$\therefore \quad \vec{\omega}_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \vec{\omega}_B$$

$$\vec{\omega}_I = \begin{bmatrix} 0 \\ 0 \\ V/R \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ V/R \end{bmatrix} = \vec{\omega}_B = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\sin\theta\, V/R \\ \cos\theta\, V/R \end{bmatrix}$$

6(b)

~~Classical Dynamics~~

Let $1 \rightarrow$ roll axis, $2 \rightarrow$ yaw ~~pitch axis~~ ~~yaw axis~~, $3 \rightarrow$ pitch ~~yaw~~ ~~pitch~~ axis

Euler's equation in the roll axis frame,
(principal axes)

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 + M_1$$

$\dot{\omega}_1 = 0$ (steady flight)

$$\therefore \quad M_1 = -(I_2 - I_3)\,\omega_2 \omega_3$$

$$= -(C - A)\left(-\sin\theta \cos\theta \, \frac{V^2}{R^2}\right)$$

$$M_1 = \frac{1}{2}(C - A)\sin 2\theta \, \frac{V^2}{R^2}$$

External moment about roll axis

Ch. L. Rama Rao

(64) Given the principle moments of inertia 500, 600 & 300 Kgm^2.

satellite initially spins about minor axis with $\omega_1 = 0.2$ rad/s

After the masses pull out, moment of inertia changes

Therefore applying conservation of angular momentum

{∴ no external torque}

$$I_1\omega_1 = I_2\omega_2$$

$$(300 \times .2) = (300 + 2 \times 1 \times (50)^2)\,\omega_2$$

{assuming that initially the masses were very close to the minor axis}

$$\Rightarrow \omega_2 = \frac{60}{5300}$$

$$= .0113 \text{ rad/s}$$

∴ The intended spin rate = .0113 rad/s

This scheme will not work as designed. ~~Since there are flexible antennae there will be energy dissipation.~~

~~Angular momentum still remains conserved as the forces are internal.~~

~~The body goes to states of lower K.E & lowest K.E is along major axis. On the other hand its max along minor axis.~~

~~Therefore during the process, the satellite will tend towards major axis spin.~~

Lets call the axis before the change I_1, I_2 & I_3. I_1 being the major & I_3 the minor axis.

~~classes~~

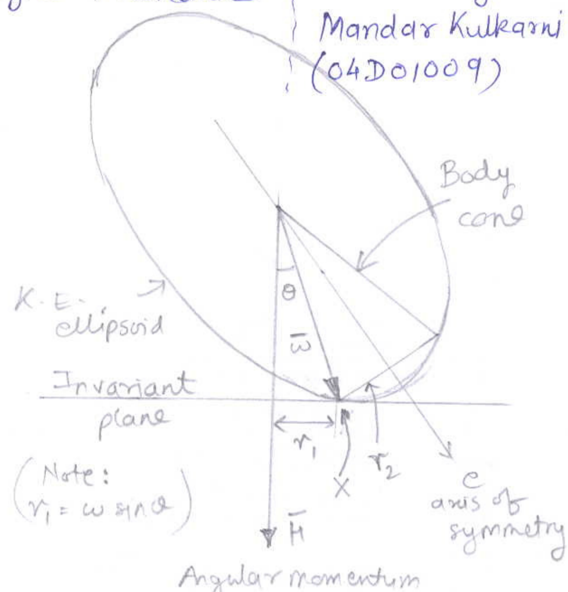
After this change, I_1 still remains the major axis. however I_3 becomes intermediate & I_2 the minor axis.

Now, since spin about Intermediate is unstable during the transition the satellite will change its spin & finally end up spinning along major axis.

AE 415: Space flight Mechanics

By
Mandar Kulkarni
(04D01009)

Problem (68)



To prove:
Axis of symmetry of the body precesses at a rate $\omega_p = \frac{H}{A}$.

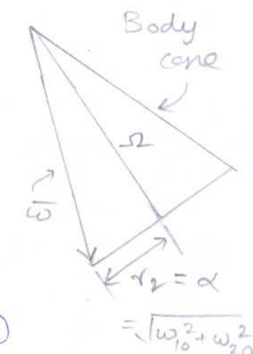
Given: $I_{xx} = I_{yy} = A$,
$I_{zz} = C$

As a part of the previos problem (67), it can be shown that $(\bar{\omega})_B$ is of the form

$$(\bar{\omega})_B = \begin{bmatrix} \alpha \sin\beta \\ \alpha \cos\beta \\ \Omega \end{bmatrix}$$

And $\bar{\omega}$ traces a body cone of radius

$r_2 = \alpha = \sqrt{\omega_{10}^2 + \omega_{20}^2}$, $\rightarrow$ (1)



at a coning rate: $\omega_c = \frac{\Omega (C-A)}{A}$ $\rightarrow$ (2)

Now consider a point X which is the instantaneous point of contact between the K.E. ellipsoid and Invariant plane.
Since K.E. ellipsoid rolls without slipping on the invariant plane, velocity at point X can be written as:

$v_X = \boxed{\omega_p \cdot r_1 = \omega_c \cdot r_2}$, where

$\rightarrow$ (3)

PTO

ω_p = angular rate at which axis of symmetry (e) precesses (about the vector $\bar{H}$)

r_1 = radius of the herpolhode, i.e. radius of the circle traced by tip of $(\bar{\omega})$ on the invariant plane

ω_c = angular rate at which $(\bar{\omega})$ traces the body cone

r_2 = radius of the polhode, i.e. radius of the circle traced by tip of $(\bar{\omega})$ on the K.E. ellipsoid.

We want to find ω_p.

Let θ be the angle between $\bar{\omega}$ and $\bar{H}$

$$(\bar{\omega})_B = \begin{bmatrix} \alpha \sin\beta \\ \alpha \cos\beta \\ \Omega \end{bmatrix} ; \quad (\bar{H})_B = I_B(\bar{\omega})_B = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \alpha \sin\beta \\ \alpha\cos\beta \\ \Omega \end{bmatrix} = \begin{bmatrix} A\alpha \sin\beta \\ A\alpha\cos\beta \\ C\Omega \end{bmatrix}$$

$$\therefore \quad \omega\cos\theta = \frac{(\bar{\omega})_B^T (\bar{H})_B}{H} = \frac{A\alpha^2 + C\Omega^2}{(A^2\alpha^2 + C^2\Omega^2)^{1/2}}$$

$$\therefore \quad \omega^2 \sin^2\theta = \omega^2 - \omega^2\cos^2\theta$$

$$= (\alpha^2 + \Omega^2) - \frac{(A\alpha^2 + C\Omega^2)^2}{(A^2\alpha^2 + C^2\Omega^2)}$$

$$= \frac{(\alpha^2+\Omega^2)(A^2\alpha^2 + C^2\Omega^2) - (A^2\alpha^4 + C^2\Omega^4 + 2AC\alpha^2\Omega^2)}{(A^2\alpha^2 + C^2\Omega^2)}$$

$$= \frac{(C^2 + A^2 - 2AC)\alpha^2\Omega^2}{H^2}$$

$$\therefore \quad r_1 = \omega\sin\theta = \sqrt{\frac{(C^2+A^2-2AC)\alpha^2\Omega^2}{H^2}} = \frac{(C-A)\alpha\Omega}{H} \longrightarrow (4)$$

$\therefore$ Putting (1), (2) and (4) in (3), we get

$$\omega_p = \frac{\omega_c r_2}{r_1} = \frac{\Omega(C-A)}{A} \times \alpha \times \frac{H}{(C-A)\alpha\Omega} = \frac{H}{A}$$

$\therefore$ Axis of symmetry precesses at a rate $\boxed{\omega_p = \frac{H}{A}}$

69) Let the principal moments of inertia of the disc be A, A and C. (A is MI about axis lying in plane of disc and C is that about axis perpendicular to the plane of disc)

$A = \frac{1}{4} m r^2$ and $C = \frac{1}{2} m r^2$ where m is mass of disc and r is radius of disc

$\therefore C = 2A$ ... (i)

Let $\bar{\omega} = [\omega_1 \ \omega_2 \ \omega_3]^T$ be the angular velocity of body frame with respect to inertial frame.

We have, from Euler's equation for rotational dynamics,

$I_B \dot{\bar{\omega}} + \Omega' I_B \omega = (\bar{M})_B$

Where I_B is the moment of inertia matrix and $(\bar{M})_B$ is the external torque in body frame.

In the absence of any external torque,

$I_B \dot{\bar{\omega}} + \Omega' I_B \omega = 0$

$$\therefore \underbrace{\begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}}_{I_B} \underbrace{\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}}_{\dot{\bar{\omega}}} + \underbrace{\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}}_{\Omega'} \underbrace{\begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}}_{I_B} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{\omega} = 0$$

This leads to system of 3 differential equations

$A \frac{d\omega_1}{dt} + \omega_2 \omega_3 (C - A) = 0$ .. (ii)

$A \frac{d\omega_2}{dt} + \omega_1 \omega_3 (A - C) = 0$ .. (iii)

$C \frac{d\omega_3}{dt} = 0$ .. (iv)

(iv) indicates that $\omega_3 =$ constant.

Solving (ii) and (iii) simultaneously, we get

$\omega_1 = \Omega \sin \alpha t$ (w/o loss of generality, we can assume

$\omega_2 = -\Omega \cos \alpha t$ phase $\phi = 0$)

where Ω is the amplitude and $\alpha = \frac{(C-A)\,\omega_3}{A}$, the angular frequency

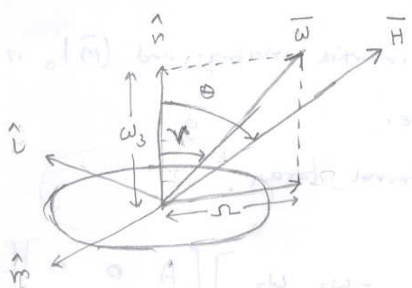
The angular momentum vector in body frame is given by

$$(\bar{H})_b = I\omega = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

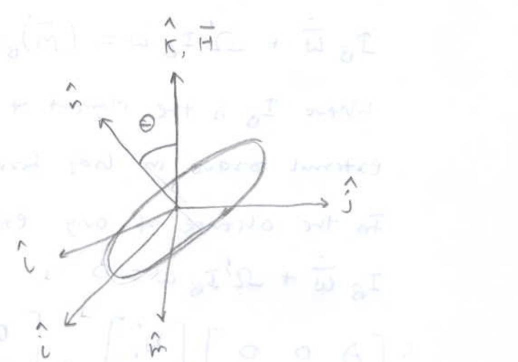
~~$\therefore (\bar{H})_b =$~~

$$\therefore (\bar{H})_b = \begin{bmatrix} A\Omega \sin \alpha t \\ -A\Omega \cos \alpha t \\ C\omega_3 \end{bmatrix} \quad \text{--- (v)}$$

Note: Ω here refers to the magnitude of ω_2 and ω_3.



Body frame



Inertial frame

Since the disc is spun about an axis which makes an angle γ with the normal, the angle b/w $\hat{n}$ and $\bar{\omega}$ is γ.

($\hat{l}, \hat{m}, \hat{n}$ are unit vectors of Body frame and $\hat{i}, \hat{j}, \hat{k}$ are unit vectors of inertial frame)

The required angle θ is the one that the normal ~~traces out in~~ makes with $\bar{H}$ in ~~body~~ for inertial frame. In inertial frame, $\bar{H}$ is constant. In body frame, $\hat{n}$ is constant.

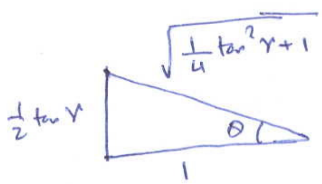
we have, $\cos\theta = \frac{\hat{n}\cdot\bar{H}}{|\bar{H}|} = \frac{C\omega_3}{\sqrt{A^2\Omega^2 + C^2\omega_3^2}}$ [from (v)] ...(vi)

$$\therefore \cos\theta = \frac{1}{\sqrt{\left(\frac{A}{C}\right)^2\left(\frac{\Omega}{\omega_3}\right)^2 + 1}}$$

From figure, $\tan\gamma = \frac{\Omega}{\omega_3}$ and we have $\frac{A}{C} = \frac{1}{2}$ [from (i)]

Putting these,

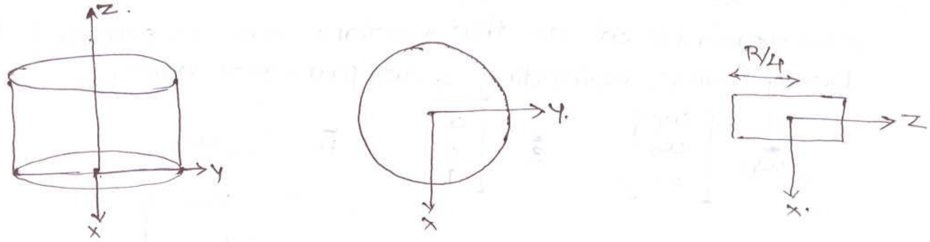
$$\cos\theta = \frac{1}{\sqrt{\frac{1}{4}\tan^2\gamma + 1}}$$



$$\therefore \boxed{\tan\theta = \frac{1}{2}\tan\gamma}$$

..(Q.E.D.)

Pr.70 AE 415: Space Flight Mechanics. Mallesh.V.B.



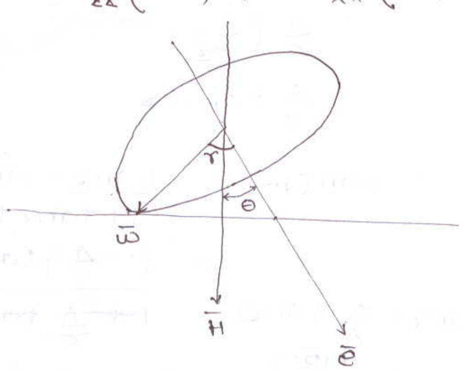
The cylindrical disc is an axis-symmetric body and it's axes are fixed as shown above. z-axis being the axis-of-symmetry is a principal axis. M.I for any orientation of x-y in xy plane are equivalent and minimum when origin is at the centre of the height of the cylinder. Hence, x, y in the above figure are principal axes too.

$$\therefore \quad I_B = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$I_{zz} = \dfrac{MR^2}{2}$ (Moment of inertia of circular cross-section about it's centre)

$I_{xx} = I_{yy} = \dfrac{M}{12}\left(\dfrac{R}{2}\right)^2 = \dfrac{MR^2}{48}$ (M.I of rod about it's centre)

$I_{zz} (= C) > I_{xx} (= A) \Rightarrow$ Oblate body.



consider oblate body.
For an inertial observer, ~~rotating about~~ axis of symmetry $\bar{e}$ precesses about $\bar{H}$, producin. cone with $\theta = 15^\circ$.

Angular velocity vector produces a cone of half angle γ about $\bar{e}$. ~~axi~~

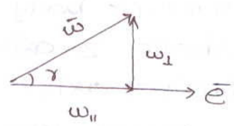
Hence, an inertial observer sees $\bar{\omega}$ produce a cone of half angle $(\gamma - \theta)$ about $\bar{H}$.
Note that, $\bar{H}$ is a fixed vector and inertial observer sees other vectors w.r.t this vector $\bar{H}$.

Thus, given $\theta = 15^\circ$, we have to find $(\gamma - \theta)$

components of all the vectors are expressed in body frame, coinciding with principal axes.

$$\bar{\omega}_B = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \qquad \bar{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \therefore \bar{H} = I_B \omega_B$$

$$= \begin{bmatrix} A\omega_1 \\ A\omega_2 \\ C\omega_3 \end{bmatrix}$$



$$\tan\gamma = \frac{\omega_\perp}{\omega_{\parallel}}$$

$$= \frac{\omega_{12}}{\omega_3}$$

$$\omega_{\parallel} = \bar{\omega} \cdot \hat{e}$$

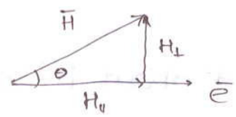
$$= \omega^T e$$

$$= \omega_3$$

$$\therefore \omega_\perp = \sqrt{\omega^2 - \omega_3^2}$$

$$= \sqrt{\omega_1^2 + \omega_2^2}$$

$$= \omega_{12}.$$



$$\therefore \tan\theta = \frac{H_\perp}{H_{\parallel}}$$

$$= \frac{A\omega_{12}}{C\omega_3}$$

$$= \frac{A}{C}\left(\frac{\omega_{12}}{\omega_3}\right)$$

$$= \frac{A}{C}\tan\gamma.$$

$$H_{\parallel} = H^T e$$

$$= C\omega_3$$

$$H_\perp = \sqrt{H^2 - H_{\parallel}^2}$$

$$= \sqrt{A^2(\omega_1^2 + \omega_2^2) + C^2\omega_3^2 - C^2\omega_3^2}$$

$$= \sqrt{A^2\omega_{12}^2}$$

$$= A\omega_{12}.$$

$$\therefore \cancel{\tan(\gamma - \theta)} = \cancel{\frac{\tan\gamma - \tan\theta}{1 \mp \tan\gamma\tan\theta}} = \cancel{\frac{(1 - \frac{A}{C})\tan\theta}{1 + \frac{A}{C}\tan}}$$

$$\therefore \tan\gamma = \frac{C}{A}\tan\theta$$

$$= \frac{\left(\frac{MR^2}{2}\right)}{\left(\frac{MR^2}{48}\right)}\tan\theta = 24\tan 15^\circ$$

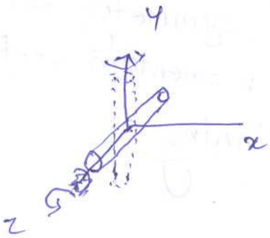
$$= 6.43$$

$$\therefore \gamma = 81.16^\circ$$

$$\therefore \underline{\underline{\gamma - \theta = 66.16^\circ}}.$$

Nikhil Bhat 04D01013

75) An axisymmetric body is spin stabillized about [0 0 1], it is to be re-oriendted such that it rotates about [0 1 0]



In the principal body frame

$I_{zz} = C$ & $I_{xx} = I_{yy} = A$

Initial angular ~~momentum~~ velocity $= [0\ 0\ \Omega]$

Initial angualar momentum $= [0\ 0\ C\Omega]$

(i) ~~Imart~~ Impart an angular momentum ~~trus~~ thrust

$\Delta H = [0\ \ C\Omega\ \ 0]$

This will change the total angular momentum to $[0\ C\Omega\ C\Omega]$

say $H_1 = [0\ C\Omega\ \ C\Omega]$

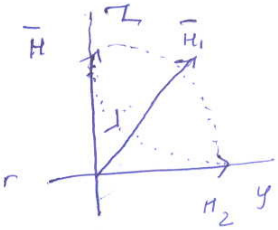
Now the body will start precessing about the new direction of angular momentum.

The rate of precession is given by $\frac{|H|}{A} = \frac{\sqrt{2}\, C\Omega}{A}$

After precession of π rad the body axis will now be aligned with y axis

Time taken for the alignment is

$= \frac{\pi A}{\sqrt{2}\, C\Omega}$



At this moment we will perform our second maneauver

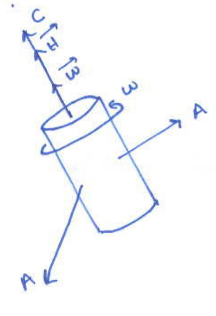
(ii) Impart $\Delta H = [\cancel{0 \; c\Omega \; 0}] \; [0 \; 0 \; -c\Omega]$

at $t = \frac{\pi A}{\sqrt{2} c \Omega}$

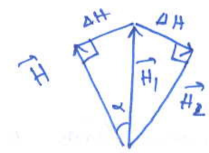
After ~~the~~is maneuver the axis of symmetry will ~~again~~ be aligned with angular momentum vector and the body will continue spinning about [0 1 0]

Spaceflight Dynamics.
[Vishal Prabhu 04d11019]

Problem 76)



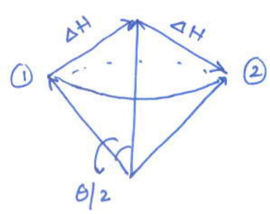
$\vec{\Delta H}$ has to be orthogonal to H



$|\vec{H}| = |\vec{H_2}|$

Also, $|\Delta H| = |H| \tan \alpha$.

- Single step, 2 impulse maneuver:



Total impulse required

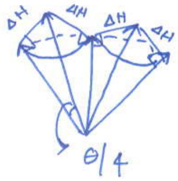
$= |\Delta H_T|_1 = 2|\Delta H| = 2 H \tan \frac{\theta}{2}$

The time required is the time for the body to precess from ① to ② i.e. $\phi = \pi$.

From problem 68) $\dot{\phi} = \frac{H}{A}$

$\therefore$ time $t_1 = \frac{\pi}{\dot{\phi}} = \frac{\pi A}{H}$

- Two step, 4 impulse maneuver:



Total impulse =

$|\Delta H_T|_2 = 4|\Delta H| = 4 H \tan \frac{\theta}{4}$

The time $t_2 = 2\frac{\pi}{\dot{\phi}} = \frac{2\pi A}{H}$.

Now $\cancel{4\tan 2\tan\frac{\theta}{4} = \frac{4\tan\theta/4}{1-\tan^2\theta/4} \Leftarrow 4\tan\theta/4}$

$$\tan 2\left(\frac{\theta}{4}\right) = \frac{2\tan \theta/4}{\underbrace{1-\tan^2\theta/4}_{\leq 1}}$$

$$\Rightarrow \quad 4H\tan\frac{\theta}{4} \leq 2H\tan\frac{\theta}{2} \quad \Rightarrow \quad |\Delta H_T|_2 < |\Delta H_T|_1.$$

Also, $t_2 > t_1$.

$\therefore$ 2 step requires more time but less total impulse.

57 Let the angular velocity components in body frame be $(\omega_1, \omega_2, \omega_3)$

04001008
Gaurav Kattar

Now for any vector $\vec{r}$ fixed in B,

$$\boxed{(\dot{\vec{r}})_I = \dot{R}\,(r)_B = R\,\Omega(r)_B} \quad \text{—— (1)}$$

where R – is the rotation matrix which relates as follows

$$\boxed{(\vec{r})_I = R\,(\vec{r})_B}$$

Ω – is the matrix $\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$ where

$(\omega_1, \omega_2, \omega_3)$ are the ~~body~~ components of ang. vel. of B frame w.r.t. I frame, as projected onto the B frame.

Now multiplying both sides of (1) by $R^{-1} = R^T$, we get

$$(\dot{\vec{r}})_B = \Omega\,(\vec{r})_B$$

Hence if $\vec{r} = \hat{i}$ vector of B frame,

$$\begin{bmatrix} 0 \\ 1 \\ -2.4 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \omega_3 = 1, \quad \omega_2 = 2.4$

Similarly for $\vec{r} = \hat{j}$ (of B frame)

$$\begin{bmatrix} -1 \\ 0 \\ 3.1 \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad w_3 = 1$ and $w_1 = 3.1$.

$\Rightarrow \quad w_1 = 3.1, \quad w_2 = 2.4, \quad w_3 = 1$

and $(\vec{w})_B = (3.1, 2.4, 1)$.