If x is a Gaussian random vector, then $\bar{x} = \mu$, $P_{xx} = P$.

<u>Fact.</u> $Pr.\left((x-\mu)^T P^{-1}(x-\mu) \le R^2\right) = 1 - e^{-R^2/2}$.

What kind of a set does the inequality $(x-\mu)^T P^{-1}(x-\mu) \le R^2$ represent?

<u>Ex.</u> $\mu = 0$, $P = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$.

$$(x-\mu)^T P^{-1}(x-\mu) = \frac{x_1^2 + x_2^2}{\sigma^2}.$$

$\therefore$ The set $(x-\mu)^T P^{-1}(x-\mu) \le R^2$ is the disc. $\{x : x_1^2 + x_2^2 \le \sigma^2 R^2\}$.

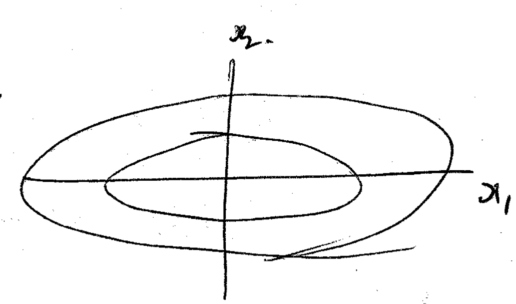
Can find the size of the disc within which x can be guaranteed to lie with a given probability.

Ex. $\mu = 0$, $P = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$.

$$(x-\mu)^T P^{-1}(x-\mu) = \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}.$$

$\therefore$ The set $\{(x-\mu)^T P^{-1}(x-\mu) \le R^2\}$ is the interior of the ellipse

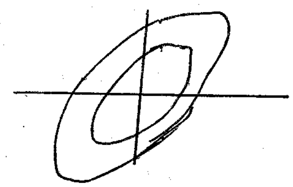
$$\left\{ x : \frac{x_1^2}{R^2\sigma_1^2} + \frac{x_2^2}{R^2\sigma_2^2} = 1 \right\}.$$



Note: semiaxes $\propto$ variances.

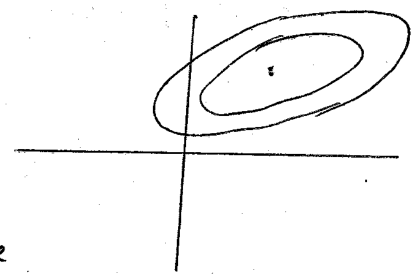
__Ex__ $\mu = 0$, P not diagonal.

The eqn. $(x-\mu)^T P^{-1}(x-\mu) = R^2$ describes a family of concentric ellipses, not necessarily aligned with the coordinate axes.



__Ex__. $\mu \neq 0$ P not diagonal.

The eqn. $(x-\mu)^T P^{-1}(x-\mu) = R^2$ describes a family of ellipses centered at (μ_1, μ_2).



Thus, for a Gaussian random vector, the covariance matrix gives a direct measure of how the values of the random vector are spread out in 2D.

How do mean & covariance matrix change under linear transformations?

Suppose random vectors y & x are related by

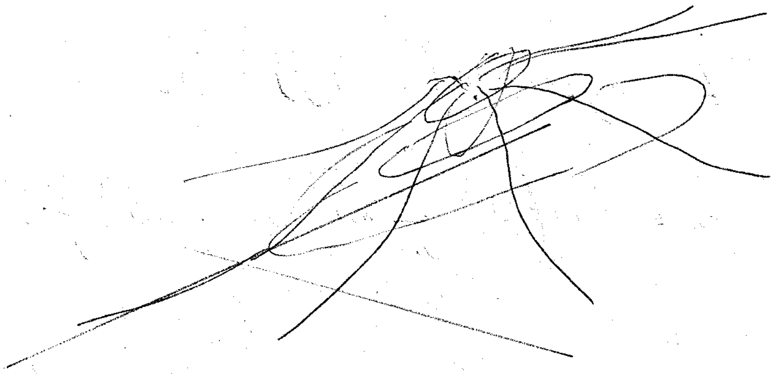
$$y = Ax + b, \quad \text{where } A \text{ is a const. } 2\times 2 \text{ matrix},$$

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Then $\cancel{E(y)} = \bar{y} = A\bar{x} + b$.

$$P_{yy} = A P_{xx} A^T.$$

Plot of the 2D Gaussian pdf.



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All the notions extend to higher dimensional random vectors.

Analysis of Navigational Error:

In the absence of measurement noise or error, the measurement y & position x are related by

$$y = g(x). \qquad (1)$$

In the presence of an additive random measurement error δy, the actual measured value is related to position x by

$$y = g(x) + \delta y. \qquad (2)$$

When (1) is used to solve for position x using the erroneous measurement (2), the resulting ~~error δx satisfies~~ position estimate x_0 satisfies

$$g(\cancel{x+\delta x}\; x_0) = y = g(x) + \delta y.$$ The error is $\delta x = x_0 - x$.

Retaining terms to first order yields

$$\frac{\partial g}{\partial x}(x)\,\delta x = \delta y, \quad \text{where } \frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}.$$

$\therefore\ \delta x = \left(\dfrac{\partial g(x)}{\partial x}\right)^{-1} \delta y.$

If $\delta y \sim N(0, P_{yy})$, then

$$\delta x \sim N\left(0, \frac{\partial g}{\partial x}(x)^{-1} P_{yy} \frac{\partial g}{\partial x}(x)^{-T}\right).$$

If position x is expressed in terms of Cartesian coordinates, then $\delta x^T \delta x$ is the square error between the actual position & the calculated position.

① $\therefore$ Mean square error

$$= E(\delta x^T \delta x) = E[\mathrm{tr.}(\delta x^T \delta x)] = E[\mathrm{tr.}\, \delta x\, \delta x^T]$$

$$= \mathrm{tr.}\, E[\delta x\, \delta x^T] = \mathrm{tr.}\, P_{xx} = \mathrm{tr.} \frac{\partial g}{\partial x}(x)^{-1} P_{yy} \frac{\partial g}{\partial x}(x)^{-T}.$$

② Error ellipses defined by $\delta x^T P_{xx}^{-1} \delta x = R^2$.

Recall that $\Pr.(\delta x^T P_{xx}^{-1} \delta x \le R^2) = 1 - e^{-R^2/2}$ (HOFFMAN)

R.	Value of Probability $(1-e^{-R^2/2})$	Name.
1	39.4%	Standard error ellipse
1.18	50%	Circular error probable.
$\sqrt{2}$	63.2%	Distance root mean square.
2	86.5%	2σ ellipse.
2.45	95%	95% Confidence ellipse
3	98.9%	3σ ellipse.