Solutions.

1 a). The unit vector from A to the observer O is

$$e_A = \frac{1}{\|r_o - r_A\|^{\cancel{2}}}(r_o - r_A)$$

$$r_o = \begin{bmatrix} 0 \\ -2.3 \end{bmatrix}, \quad r_A = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad r_B = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} \quad r_c = \begin{bmatrix} 1.3 \\ 0 \end{bmatrix}.$$

$$\therefore e_A = \frac{1}{\sqrt{1 + 2.3^2}}\begin{bmatrix} 1 \\ -2.3 \end{bmatrix} = \begin{bmatrix} 0.3987 \\ -0.2917 \end{bmatrix}.$$

Unit vector from B to O is $e_B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Unit vector from C to O is

$$e_c = \frac{1}{\|r_o - r_c\|}(r_o - r_c) = \frac{1}{\sqrt{1.3^2 + 2.3^2}}\begin{bmatrix} -1.3 \\ -2.3 \end{bmatrix}$$

$$= \begin{bmatrix} -0.492 \\ -0.87 \end{bmatrix}.$$

The measured range rates to points A & B are given by

$\dot{\rho}_A = e_A^T v$, $\dot{\rho}_B = e_B^T v$, where v is the velocity vector of O.

Letting $A = \begin{bmatrix} e_A^T \\ e_B^T \end{bmatrix}_{2\times 2}$, it follows that

$$v = A^{-1}\begin{bmatrix} \dot{\rho}_A \\ \dot{\rho}_B \end{bmatrix}.$$

$\therefore$ estimate of velocity

$$V = \begin{bmatrix} 0.3987 & -0.917 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -4.58 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.012 \\ 5 \end{bmatrix} m/s. \qquad - 2\tfrac{1}{2} \text{ mark.}$$

The covariance matrix P_{AB} of the error vector $\begin{bmatrix} \delta\dot{\rho}_A \\ \delta\dot{\rho}_B \end{bmatrix}$ is the 1-2 submatrix of the covariance matrix of the error vector $[\delta\dot{\rho}_A, \delta\dot{\rho}_B\ \delta\dot{\rho}_C]^T$. That is,

$$P_{AB} = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}. \qquad -$$

$\therefore$ ~~covariance matrix of the~~ The error in the velocity estimate is related to the measurement error by $\delta V = A^{-1} \begin{bmatrix} \delta\dot{\rho}_A \\ \delta\dot{\rho}_B \end{bmatrix}$.

$\therefore$ covariance matrix of the estimation error is

$$P_{VV} = A^{-1} \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix} A^{-T}$$

$$= \begin{bmatrix} 1.6914 & 0.4392 \\ 0.4392 & 0.3 \end{bmatrix}. \qquad - 1 \text{ mark.}$$

The ~~R~~RMS value of the estimation error is

$$= \sqrt{\text{trace } P_{vv}} = \sqrt{1.9914} = 1.4112 \text{ m/s.}$$ — 1/2 mark

NOTE: If measurements A & C had to be used, to estimate velocity, then the estimate ~~is~~ would be given by

$$V = \begin{bmatrix} e_A^T \\ e_C^T \end{bmatrix}^{-1} \begin{bmatrix} \dot{\rho}_A \\ \dot{\rho}_C \end{bmatrix} = \begin{bmatrix} 0.0055 \\ 4.9969 \end{bmatrix}.$$

The estimate ~~two~~ error covariance matrix would be

$$P_{vv} = \begin{bmatrix} e_A^T \\ e_C^T \end{bmatrix}^{-1} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} e_A^T \\ e_C^T \end{bmatrix}^{-T} = \begin{bmatrix} 0.5678 & 0.0091 \\ 0.0091 & 0.1384 \end{bmatrix}$$

resulting in a RMS error of $\sqrt{\text{trace } P_{vv}} = 0.8404$ m/s.

If measurements B & C had to be used, then velocity estimate

$$v = \begin{bmatrix} e_B^T \\ e_C^T \end{bmatrix}^{-1} \begin{bmatrix} \dot{\rho}_B \\ \dot{\rho}_C \end{bmatrix} = \begin{bmatrix} 0.0 \\ +5.0 \end{bmatrix}$$

$$P_{vv} = \begin{bmatrix} e_B^T \\ e_C^T \end{bmatrix}^{-1} \begin{bmatrix} 0.3 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} e_B^T \\ e_C^T \end{bmatrix}^{-T} = \begin{bmatrix} 1.9708 & -0.5305 \\ -0.5305 & 0.3 \end{bmatrix}.$$

resulting in RMS error $\sqrt{\text{tr.} P_{vv}} = 1.5069$.

Note that the least RMS value results when the ~~most~~ worst sensor B is ignored. However, part (b) will show that BLUE gives least errors compared to all the 3 combinations above.

b). To use BLUE, we need to relate measurements to the ~~quanti~~ quantity that we wish to estimate & the measurement error. In this case,

$$y = H V + \delta w, \quad \text{where.}$$

$$y = \begin{bmatrix} \dot{\rho}_A \\ \dot{\rho}_B \\ \dot{\rho}_C \end{bmatrix} = \begin{bmatrix} -4.58 \\ -5 \\ -4.35 \end{bmatrix}$$ is the measurement vector.

$$H = \begin{bmatrix} e_A^T \\ e_B^T \\ e_C^T \end{bmatrix} = \begin{bmatrix} 0.3987 & -0.917 \\ 0 & -1 \\ -0.492 & -0.87 \end{bmatrix}.$$

$2\frac{1}{2}$ mark

V is the velocity vector to be estimated

& $\delta w = [\delta \dot{\rho}_A \; \delta \dot{\rho}_B \; \delta \dot{\rho}_C]$ is the measurement error.

The covariance matrix of δw is given to be

$$P_{ww} = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

∴ The best linear unbiased estimate is given by

$$\hat{V} = (H^T P_{ww}^{-1} H)^{-1} H^T P_{ww}^{-1} y$$

$$= \begin{bmatrix} 1.3213 & -0.3748 & -0.9618 \\ -0.4663 & -0.2436 & -0.3779 \end{bmatrix} y = \begin{bmatrix} 0.0066 \\ 4.9977 \end{bmatrix}.$$

1 mark

The covariance matrix of the estimate error is ②

$$P_{\tilde{x}\tilde{x}} = (H^T P_{ww}^{-1} H)^{-1}$$

$$= \begin{bmatrix} 0.5235 & -0.0197 \\ -0.0197 & 0.1197 \end{bmatrix} \quad \text{— 1 mark.}$$

The RMS value of the estimation error is

$$\sqrt{\text{trace } P_{\tilde{x}\tilde{x}}} = \sqrt{0.6432} = 0.8020 \quad \text{— } \tfrac{1}{2} \text{ mark.}$$

2. The measurements are $x_1 = 806.63$ m.

$x_2 = 807.23$ m.

The standard deviations are $\sigma_1 = 0.321$ m.

$\sigma_2 = 0.334$ m.

1 mark.

The errors are uncorrelated. $\therefore \sigma_{12} = 0$.

The minimum variance estimate is given by

$$\hat{x} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} x_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} x_2 = 806.91\text{m.} \quad \text{— 2 marks.}$$

The standard deviation of the estimation error is

$$\sigma = \sqrt{\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2+\sigma_2^2}} = 0.231\text{ m.} \quad \text{— 1 mark.}$$

3 a The errors in ~~the mea~~ the differences of bearings are related to the errors in the bearing measurements by

$$\delta\theta_{AB} = \delta\theta_A - \delta\theta_B, \quad \delta\theta_{BC} = \delta\theta_B - \delta\theta_C.$$

$\therefore$ ~~The~~ Since all bearing measurements errors have the same mean, $\delta\theta_{AB}$ & $\delta\theta_{BC}$ have zero means. 1/2 mark

Moreover, since $\delta\theta_A, \delta\theta_B, \delta\theta_C$ are uncorrelated, 1/2 mark

$$\sigma_{AB}^2 = E(\delta\theta_{AB}^2) = E\left[(\delta\theta_A - \delta\theta_B)^2\right]$$

$$= \sigma_A^2 + \sigma_B^2 = 2\sigma^2. \qquad \text{1/2 mark}$$

$$\sigma_{BC}^2 = E(\delta\theta_{BC}^2) = E\left[(\delta\theta_B - \delta\theta_C)^2\right]$$

$$= \sigma_B^2 + \sigma_C^2 = 2\sigma^2. \qquad \text{1/2 mark}$$

& $E(\delta\theta_{AB}\,\delta\theta_{BC}) = E(\delta\theta_A\delta\theta_B - \delta\theta_A\delta\theta_C - \delta\theta_B^2 + \delta\theta_B\delta\theta_C)$

$$= -\sigma^2. \qquad \text{1/2 mark}$$

$\therefore$ If $\delta\Theta = \begin{bmatrix} \delta\theta_{AB} \\ \delta\theta_{BC} \end{bmatrix}$ ~~is the~~, then

$$P_{\Theta\Theta} = \begin{bmatrix} 2\sigma^2 & -\sigma^2 \\ -\sigma^2 & 2\sigma^2 \end{bmatrix} = \begin{bmatrix} 0.18 & -0.09 \\ -0.09 & 0.18 \end{bmatrix}. \qquad \text{1/2 mark}$$

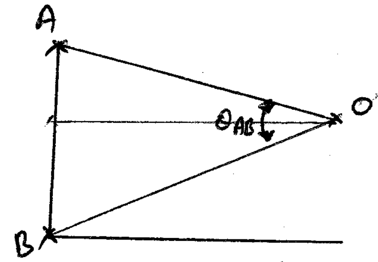
b) ~~The measure~~ The positions of the landmarks are

$$r_A = \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} 0 \\ 2.6 \end{bmatrix}, \quad r_B = \begin{bmatrix} x_B \\ y_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad r_C = \begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} 2.26 \\ 0 \end{bmatrix}$$

The estimated position of the observer is

$$r = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.54 \\ 1.34 \end{bmatrix}.$$

The measurement θ_{AB} is related to the position estimate by



$$\theta_{AB} = \tan^{-1}\frac{(y-y_B)}{(x-x_B)} + \tan^{-1}\frac{(y_A-y)}{(x-x_A)}$$

$$= 54.2^\circ$$ (numerically value is not required) | mark.

The measurement error $\delta\theta_{AB}$ & the estimate error $\delta r = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$ are related (to 1st order) by

$$\delta\theta_{AB} = \frac{\partial \theta_{AB}}{\partial x}\delta x + \frac{\partial \theta_{AB}}{\partial y}\delta y.$$

$$= \frac{1}{\|r-r_B\|^2}\left[-(y-y_B)\delta x + (x-x_B)\delta y\right]$$

$$+ \frac{1}{\|r-r_A\|^2}\left[-(y_A-y)\delta x - (x-x_A)\delta y\right].$$ — 1 mark

Substituting values yields

$$\|r-r_B\|^2 = \cancel{\sqrt{}}\,2.54^2 + 1.34^2 = 8.2472$$

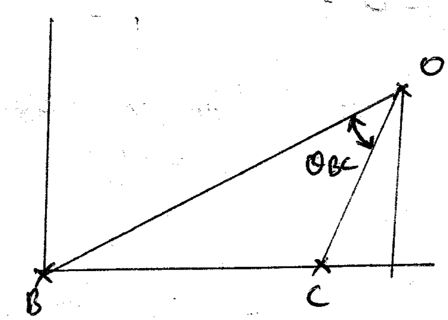
$$\|r-r_A\|^2 = 2.54^2 + (1.34-2.6)^2 = 8.0392$$

$\therefore \quad \delta\theta_{AB} = -0.319\,\delta x - 0.08\,\delta y.$

Similarly, the measurement θ_{BC} is related to estimate r by

$$\theta_{BC} = \tan^{-1}\left(\frac{x - x_B}{y - y_B}\right) - \tan^{-1}\left(\frac{x - x_C}{y - y_C}\right) \qquad \text{— 1 mark.}$$

$$= 50.38^\circ \quad \text{(not required for the answer).}$$



$$\therefore \delta\theta_{BC} = \frac{\partial\theta_{BC}}{\partial x}\,.\,\delta x + \frac{\partial\theta_{BC}}{\partial y}\,\delta y.$$

$$= \left[\frac{(y - y_B)}{\|r - r_B\|^2} - \frac{(y - y_C)}{\|r - r_C\|^2}\right]\delta x$$

$$+ \left[-\frac{(x - x_B)}{\|r - r_B\|^2} + \frac{(x - x_C)}{\|r - r_C\|^2}\right]\delta y \qquad \text{— 1 mark}$$

Substituting values yields $\|r - r_C\|^2 = (2.54 - 2.26)^2 + (1.34)^2 = 1.874$

$\therefore \delta\theta_{BC} = -0.552\,\delta x - 0.158\,\delta y$

Thus $\delta\Theta = \begin{bmatrix} \delta\theta_{AB} \\ \delta\theta_{BC} \end{bmatrix} = A\,\delta r$, where — 1 mark

$$A = \begin{bmatrix} -0.319 & -0.08 \\ -0.552 & -0.158 \end{bmatrix} \ \& \ \delta r = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}.$$

③

$\therefore \delta r = A^{-1} \delta \Theta$.

To find A^{-1}, we compute $\det A = 0.046$

$$\therefore A^{-1} = \begin{bmatrix} \frac{-0.158}{0.046} & \frac{0.008}{0.046} \\ \frac{0.552}{0.046} & \frac{-0.319}{0.046} \end{bmatrix} = \begin{bmatrix} -3.43 & 0.174 \\ 12 & -6.93 \end{bmatrix}.$$

$$\therefore P_{rr} = A^{-1} P_{\Theta\Theta} A^{-T} = \begin{bmatrix} -3.43 & 0.174 \\ 12 & -6.93 \end{bmatrix} \begin{bmatrix} 0.18 & -0.09 \\ -0.09 & 0.18 \end{bmatrix} \begin{bmatrix} -3.43 & 12 \\ 0.174 & -6.93 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2379 & -9.9739 \\ -9.9739 & 49.886 \end{bmatrix}.$$ — 2 marks.

Thus the covariance matrix of the error in the position estimate is P_{rr} as above.

4. The mean of x_1 is

$$E(x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f_{x_1 x_2}(x_1, x_2)\, dx_1 dx_2 = \int_0^1\int_0^1 x_1\, dx_1 dx_2$$

$$= \int_0^1 \frac{x_1^2}{2}\Big|_0^1 dx_2 = \tfrac{1}{2}.$$

Similarly, $E(x_2) = \tfrac{1}{2}$.

$$\therefore \bar{x} = E(x) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$ $1\tfrac{1}{2}$ mark.

The variance of x_1 is

$$\sigma_{x_1}^2 = E(x_1-\bar{x}_1)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1-1/2)^2 f_{x_1,x_2}(x_1,x_2)\,dx_1\,dx_2$$

$$= \int_0^1\int_0^1 (x_1-1/2)^2\,dx_1\,dx_2 = \frac{1}{3}(x_1-1/2)^3\Big|_0^1$$

$$= \frac{1}{3}\left[1/8-(-1/8)\right] = \frac{1}{12}.$$

| 1/2 mark

Similarly $\sigma_{x_2}^2 = E(x_2-\bar{x}_2)^2 = 1/12.$

$$\sigma_{x_1x_2} = E\left[(x_1-\bar{x}_1)(x_2-\bar{x}_2)\right] = \int_0^1\int_0^1 (x_1-1/2)(x_2-1/2)\,dx_1\,dx_2$$

$$= \left[\int_0^1 (x_2-1/2)\,dx_2\right]\left(\frac{x_1^2}{2}-\frac{x_1}{2}\right)\Big|_0^1 = 0.$$ | 1 mark.

$\therefore$ Covariance matrix $P_{xx} = \begin{bmatrix} 1/12 & 0 \\ 0 & 1/12 \end{bmatrix}.$

5) Taking Cuddalore & Port Blair to be lying on the same latitude, the rhumbline distance between the two is

$$R_{RHUMB} = R\cos\varphi\,|\lambda_B-\lambda_A|.$$

$$= 6378\times\cos 11.7\times(92.5-79.77)\times\frac{\pi}{180}\ \text{km}.$$

$$= 1387.62\ \text{km}.$$ | 1/2 mark.

The great circle distance between the two parts is given by

$$R_{GC} = R\cos^{-1}\left[\cos^2\varphi\cos(\lambda_A-\lambda_B)+\sin^2\varphi\right].$$

$$= 6378\times\frac{\pi}{180}\times\cos^{-1}\left[\cos^2 11.7\cos(92.5-79.77)+\sin^2 11.7\right]\text{ km.}$$

$$= 1387.5\text{ km.}$$

$1\frac{1}{2}$ mark.