① 

**5a** $l_{PQ} = c(t_P - t_Q)$ $\therefore$ $\delta l_{PQ} = c(\delta t_P - \delta t_Q)$

$$\therefore E(\delta l_{PQ}^2) = c^2 E(\delta t_P^2 + \delta t_Q^2 - 2\delta t_P \delta t_Q) = c^2(\sigma_P^2 + \sigma_Q^2) = 0.2925$$

$$E(\delta l_{QR}^2) = c^2(\sigma_Q^2 + \sigma_R^2) = 0.765$$

$$E(\delta_{RS}^2) = c^2(\sigma_R^2 + \sigma_S^2) = 0.765$$

$$E(\delta_{RT}^2) = c^2(\sigma_R^2 + \sigma_T^2) = 0.9225$$

$$E(\delta l_{PQ}\,\delta l_{QR}) = c^2 E[(\delta t_P - \delta t_Q)(\delta t_Q - \delta t_R)] = -c^2\sigma_Q^2 = -0.2025$$

$$E(\delta l_{PQ}\,\delta l_{RS}) = c^2 E[(\delta t_P - \delta t_Q)(\delta t_R - \delta t_S)] = 0$$

$$E(\delta l_{PQ}\,\delta l_{RT}) = c^2 E[(\delta t_P - \delta t_Q)(\delta t_R - \delta t_T)] = 0$$

$$E(\delta l_{QR}\,\delta l_{RS}) = c^2 E[(\delta t_Q - \delta t_R)(\delta t_R - \delta t_S)] = -c^2\sigma_R^2 = -0.5625$$

$$E(\delta l_{QR}\,\delta l_{RT}) = -c^2\sigma_R^2 = -0.5625$$

$$E(\delta l_{RS}\,\delta l_{RT}) = c^2 E[(\delta t_R - \delta t_S)(\delta t_R - \delta t_T)] = c^2\sigma_R^2 = 0.5625$$

$\therefore$ covariance matrix of measurement error vector is

$$\begin{bmatrix} \delta l_{PQ} \\ \delta l_{QR} \\ \delta l_{RS} \\ \delta l_{RT} \end{bmatrix} \quad P_{ll} = \left[\begin{array}{cc|cc} 0.2925 & -0.2025 & 0 & 0 \\ -0.2025 & 0.765 & -0.5625 & -0.5625 \\ \hline 0 & -0.5625 & 0.765 & 0.5625 \\ 0 & -0.5625 & 0.5625 & 0.9225 \end{array}\right]$$

$$= \begin{bmatrix} P_{l_1} & P_{l_1 l_2} \\ P_{l_1 l_2}^T & P_{l_2} \end{bmatrix}. \qquad \delta l_1 = \begin{bmatrix} \delta l_{PQ} \\ \delta l_{RS} \end{bmatrix}, \quad \delta l_2 = \begin{bmatrix} \delta l_{RS} \\ \delta l_{RT} \end{bmatrix}.$$

$$P_{l_1 l_2} = E[\delta l_1 \delta l_2^T].$$

b). For the 1st observation, $r_1 = [1000.67 \quad 878.39]^T$ ②

$r_P = [0 \quad 0]^T \quad r_Q = [0 \quad 1034]^T \quad \cancel{r_R = [1156 \quad 0]^T}$

$r_R = [1156 \quad 0]^T$

$$l_{PQ} = \|r_1 - r_P\| - \|r_1 - r_Q\| = 318.81$$

$$l_{QR} = \|r_1 - r_Q\| - \|r_1 - r_R\| = 120.678$$

Assuming small errors, we have

$$\begin{bmatrix} \|r_1-r_P\|^{-1}(r_1-r_P)^T - \|r_1-r_Q\|^{-1}(r_1-r_Q)^T \\ \|r_1-r_Q\|^{-1}(r_1-r_Q)^T - \|r_1-r_R\|^{-1}(r_1-r_R)^T \end{bmatrix} \delta r_1 = \begin{bmatrix} \delta l_{PQ} \\ \delta l_{QR} \end{bmatrix} = \delta l_1$$

thus $A_1 \delta r_1 = \delta l_1$, where $A_1 = \begin{bmatrix} -0.2366 & 0.8132 \\ 1.1622 & -1.1383 \end{bmatrix}$

$$\therefore \delta r_1 = A_1^{-1} \delta l_1 = \begin{bmatrix} 1.6844 & 1.2034 \\ 1.7198 & 0.3501 \end{bmatrix}.$$

The covariance matrix of $\delta l_1$ is

$$P_{l_1} = \begin{bmatrix} E(\delta l_{PQ}^2) & E(\delta l_{PQ}\delta l_{QR}) \\ E(\delta l_{PQ}\delta l_{QR}) & E(\delta l_{QR}^2) \end{bmatrix} = \begin{bmatrix} 0.2925 & -0.2025 \\ -0.2025 & 0.765 \end{bmatrix}$$

$\therefore$ Covariance matrix of $\delta r_1$ is

$$P_{11} = A_1^{-1} P_{l_1} A_1^{-T} = \begin{bmatrix} 1.1164 & 0.6308 \\ 0.6308 & 0.7147 \end{bmatrix}.$$

2nd observation: $r_2 = [1001.07 \quad 877.76]^T$. (3)

$r_R = [1156 \quad 0]^T \qquad r_S = [2022, 1179]^T \quad r_T = [1126 \quad 2163]^T$

$l_{RS} = \|r_2 - r_R\| - \|r_2 - r_S\| = -173.112$

$l_{RT} = \|r_2 - r_R\| - \|r_2 - r_T\| = -399.962$

Assuming small errors, we have

$$\underbrace{\begin{bmatrix} \|r_2 - r_R\|^{-1}(r_2 - r_R)^T - \|r_2 - r_S\|^{-1}(r_2 - r_S)^T \\ \|r_2 - r_R\|^{-1}(r_2 - r_R)^T - \|r_2 - r_T\|^{-1}(r_2 - r_T)^T \end{bmatrix}}_{A_2} \delta r_2 = \begin{bmatrix} \delta l_{RS} \\ \delta l_{RT} \end{bmatrix} = \delta l_2 .$$

Thus $A_2 \delta r_2 = \delta l_2$, where $A_2 = \begin{bmatrix} 0.7853 & 1.2677 \\ -0.0771 & 1.98 \end{bmatrix}$.

$\therefore \delta r_2 = A_2^{-1} \delta l_2 = \begin{bmatrix} 1.1981 & -0.7671 \\ 0.0466 & 0.4752 \end{bmatrix} \delta l_2$.

$\therefore$ Covariance matrix of $\delta l_2$ is $\quad P_{l_2} = \begin{bmatrix} 0.765 & 0.5625 \\ 0.5625 & 0.9225 \end{bmatrix}$.

$$P_{22} = A_2^{-1} P A_2^{-T}$$

$$= \begin{bmatrix} 0.6070 & 0.0066 \\ 0.0066 & 0.2349 \end{bmatrix}.$$

The combined error vectors are related by

$$\delta r = A^{-1}\delta l, \quad \text{where} \quad \delta r = \begin{bmatrix} \delta r_1 \\ \delta r_2 \end{bmatrix}, \quad \delta l = \begin{bmatrix} \delta l_1 \\ \delta l_2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix}.$$

$\therefore$ Covariance matrix of the combined measurement error is

$$P_{rr} = A^{-1} P_{ll} A^{-T} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix} \begin{bmatrix} P_{l_1} & P_{l_1 l_2} \\ P_{l_1 l_2}^{T} & P_{l_2} \end{bmatrix} \begin{bmatrix} A_1^{-T} & 0 \\ 0 & A_2^{-T} \end{bmatrix}$$

$$= \begin{bmatrix} A_1^{-1} P_{l_1} P_1^{-T} & A_1^{-1} P_{l_1 l_2} A_2^{-T} \\ A_2^{-1} P_{l_1 l_2}^{T} A_1^{-T} & A_2^{-1} P_{l_2} A_2^{-T} \end{bmatrix}.$$

$$= \left[\begin{array}{cc|cc} 1.1164 & 0.6308 & -0.2917 & -0.3532 \\ 0.6308 & 0.7147 & -0.0849 & -0.1028 \\ \hline -0.2917 & -0.0849 & 06070 & 0.0066 \\ -0.3532 & -0.1028 & 0.0066 & 0.2349 \end{array}\right].$$

$$= \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{T} & P_{22} \end{bmatrix}.$$

1c) The estimate is related to the actual position $r$ by

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix} r + \begin{bmatrix} \delta r_1 \\ \delta r_2 \end{bmatrix}.$$

$\underset{\text{estimates}}{\uparrow}$ $\qquad \underset{H.}{\uparrow}$

Hence the best linear unbiased estimate of $r$ given the estimates $r_1, r_2$ is given by

$$\hat{r} = \left(H^T P_{rr}^{-1} H\right)^{-1} H^T P_{rr}^{-1} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

$$\left(H^T P_{rr}^{-1} H\right)^{-1} H^T P_{rr}^{-1} = \begin{bmatrix} 0.6231 & -0.5009 & 0.3769 & 0.5009 \\ 0.0348 & 0.26 & -0.0348 & 0.74 \end{bmatrix}$$

$$\therefore \hat{r} = \begin{bmatrix} 0.6231 & -0.5009 & 0.3769 & 0.5009 \\ 0.0348 & 0.26 & -0.0348 & 0.74 \end{bmatrix} \begin{bmatrix} 1000.67 \\ 878.39 \\ 1001.07 \\ 877.76 \end{bmatrix} = \begin{bmatrix} 1001.128 \\ 877.9 \end{bmatrix}.$$

The root mean square error in the 1st estimate is

$$\sqrt{\operatorname{tr} P_{11}} = \sqrt{1.8311} = 1.353.$$

The root mean square error in the 2nd estimate is

$$\sqrt{\operatorname{tr} P_{22}} = \sqrt{0.8419} = 0.9175$$

The covariance matrix of the BLUE estimate is

$$P_{\hat{r}\hat{r}} = \left(H^T P_{rr}^{-1} H\right)^{-1} = \begin{bmatrix} 0.0928 & -0.0484 \\ -0.0484 & 0.1346 \end{bmatrix}.$$

(6)

Hence RMS error in the BLUE estimate is

$$\sqrt{\text{trace } P_{\hat{r}\hat{r}}} = \sqrt{0.2274} = 0.4768$$