AE 459/770 — Classical Dynamics, Spring 2008

Question Bank

Constraints

1. A simple pendulum of length l is is attached at a point that moves along the horizontal x-axis with a displacement $x_0(t) = A\sin\omega t$. Write down the constraint equation in terms of the Cartesian coordinates of the particle. Using the angle θ of the pendulum (measured counterclockwise from the negative y-axis) as a generalized coordinate, write down the transformation relating θ and the Cartesian coordinates.

2. *i)* A disk of radius r is constrained to remain vertical at all times. What is the configuration space of the disk? Is the constraint holonomic or nonholonomic? Suggest suitable independent generalized coordinates for the disk.
   ii) Repeat *i)* above in the case where the disk is also constrained to remain on the horizontal XY plane.
   iii) The disk in *ii)* above is further constrained to rotate without slipping on the XY plane. Express the constraints in terms of the generalized coordinates you suggested in *ii)* above. If there are any velocity constraints, do they restrict the configurations that the disk can possibly attain? Support your answer with appropriate analysis of the forms representing the constraints. If your answer is no, can you suggest how one might take the disk from a given initial configuration to a final desired one?
   iv) Repeat *ii)* above if the disk is further constrained to roll without slipping along the X axis only.

3. Two wheels of radius r are joined by an axle of length l. The wheels rotate independently of each other and move without slipping on a horizontal plane. Write down a set of independent generalized coordinates for this system. Write down position and velocity constraints if any. Do the velocity constraints restrict attainable configurations? Support your answers with arguments and/or analysis.

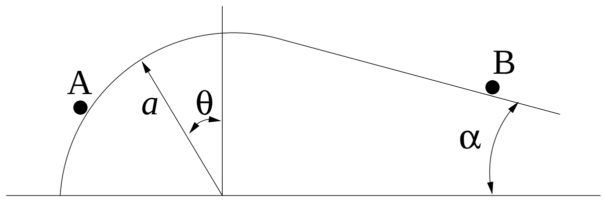
4. A dumbbell consists of two particles of mass m connected by a massless rod of length l. The dumbbell moves without friction on a horizontal plane. A knife-edge constraint at one (and only one) of the particles restricts the velocity of the particle to make an angle of 45° with the connecting rod. What is the configuration space of the system? Suggest generalized coordinates for the system. Write down the constraints acting on the system in terms of the generalized coordinates you have chosen, and state whether the constraints constrain the configurations of the system. If you assert that the velocity constraints, if any, do not constrain positions, then describe how one may take the system from any arbitrary initial configuration to an arbitrary final one. If you assert that the velocity constraints do constrain the positions, then support your answers with analysis or examples.

5. Two particles, each of mass m, can slide on the horizontal xy-plane. Particle 1 is attached to a rigid rod, and carries a knife edge parallel to the rod which constrains the velocity of particle 1 to lie along the rod. Particle 2 slides without friction on the rod and is connected to particle 1 by a spring of stiffness k and unstretched length l_0. What is the configuration space of the system (1 mark)? Suggest suitable generalized coordinates for this system (1 mark). Write down all the position and velocity constraints that are present (1.5 mark). Do the velocity constraints, if any, restrict attainable configurations? Explain briefly

6. Two particles A and B are connected by a massless rod of length l. Particle A is constrained to move along the horizontal axis, while particle B is constrained to move along the vertical axis. A and B have masses $2m$ and m, respectively. Derive the equation of motion of the system using Newton's laws and eliminating the constraint forces.

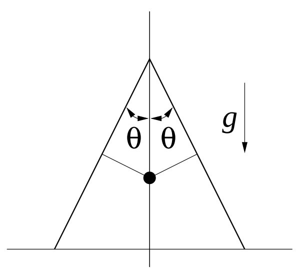
Principle of Virtual Work and d'Alembert's Principle

7. Two particles A and B having masses m_{A} and m_{B}, respectively, are connected by an axially stiff and massless cord of length L. The particles (as well as the cord) slide without friction on a ramp that has a circular segment and a straight inclined segment tangential to the circular segment as shown in the figure below. Find the angle θ at equilibrium.



8. A uniform rod of weight mg is placed in a uniform gravitational field. Show that the virtual work done by gravitational forces along any virtual displacement of the system is given by $-mg\delta z_{\mathrm{cm}}$, where z_{cm} denotes the z-component (+ve upwards) of the position of the center of mass of the rod. (Hint: Express the virtual displacement of every particle on the rod in terms of any two particles on it. Calculate the virtual displacement of each mass particle on the rod and integrate to obtain the total virtual work.)

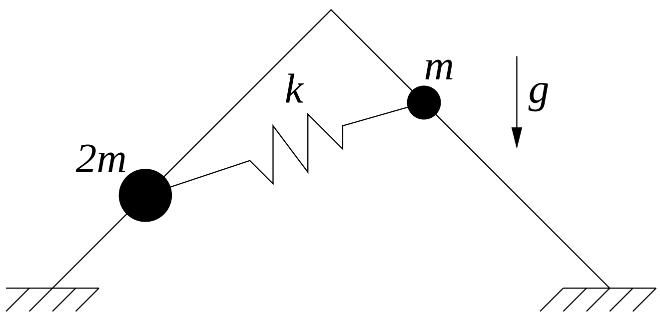
9. Two thin uniform rods of mass m and length l are pinned together at their upper ends. The lower ends slide without friction on a horizontal surface. A particle of mass m is suspended by two massless strings, each of length $l/4$ and each connected to the midpoints of the rods as shown in the figure below. Assuming planar motion, find the equilibrium values of θ in the interval $0 \le \theta < \pi/6$. (Use the conclusion of the previous problem.)



10. A rod of length l and mass m is fixed to the origin by a frictionless ball–and–socket joint. The rod is free to rest on the inner edge of the ellipse described by the equations $a^{-2}x^2 + b^{-2}y^2 = 1$, $z = c$, where $a > b > 0$ and $a^2 + c^2 < l^2$. Assuming no friction, find all equilibrium positions of the rod. (Use the conclusion of Problem 7.)

11. Two particles having masses $2m$ and m slide under gravity without friction on two rigid rods inclined at 45° with the horizontal as shown in the figure below. The two particles are connected by a linear spring of stiffness k. Use the principle of virtual work to solve

for the spring force F and the inclination θ of the spring in the equilibrium configuration.



12. A string of length $2L$ is attached at fixed points A and B that are seperated by a distance L. A particle of mass m slides without friction on the string which remains taut. Write down all the constraints on the particle. Use the Principle of Virtual Work to locate all equilibrium points of the particle.

13. A double pendulum consists of two simple pendula of masses m_1 and m_2, and lengths l_1 and l_2 connected end to end. The connecting links are massless. Motion takes place in a vertical plane. A horizontal force of magnitude $+P$ acts on the outboard mass. Using any of the methods discussed in class, find the equilibrium configurations of the system.

14. Show that, for a scleronomic system of n particles, the instantaneous rate at which external forces do work on the system is given by $Q^{\mathrm{T}}\dot{q}$, where Q is the vector of generalised forces and $\dot{q}$ is the vector of generalised velocities for some set of generalised coordinates. Does the same statement hold for a rheonomic system? (Hint: The statement is known to be true in Cartesian coordinates.)

15. A particle moving in 3D space is subject to the workless velocity constraint $(y-z)\dot{x} + (z-x)\dot{y} + (x-y)\dot{z} = 0$. Is this constraint exact, integrable, or neither? The particle is also subjected to a force field given by $\mathbf{F} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$. Use the principle of virtual work to find all possible equilibrium positions of the particle.

16. A particle of mass m is suspended from a point by an inextensible string, and moves under uniform gravity acting along the negative z-axis. Assuming that the string remains taut, use d'Alembert's principle to show that the motion of the particle satisfies $z\ddot{y} - y\ddot{z} = yg$ and $z\ddot{x} - x\ddot{z} = xg$, (where the origin of the coordinate frame is chosen to be the point of suspension).

Lagrange's Equations: Independent Generalized Coordinates

17. Use Lagrange's equations to show that a particle constrained to move on a circular cylinder without friction traces out a helix, a circle or a straight line in the absence of gravity (and other applied forces).

18. Use Lagrange's equations to show that every motion of a particle that is constrained to move along a two-dimensional sphere without friction and without any external forces traces out a great circle on the sphere.

19. A mass m restricted to move in a vertical plane is suspended in a uniform gravitational field from a fixed support by a spring of stiffness k. Use the Lagrangian method to write down the equations of motion of the mass in terms of suitably chosen polar coordinates.

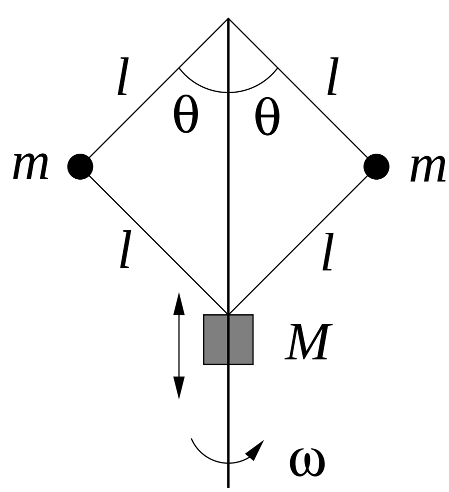
20. Derive the equations of motion of a double pendulum consisting of two simple pendula connected end to end. Assume that the connecting links are rigid and massless and motion takes place in a vertical plane under gravity.

21. A particle of mass m is connected by a massless spring of stiffness k and unstretched length r_0 to a point that is moving along a circular path of radius a at a uniform angular rate of ω. Find the equations of motion assuming that the particle moves without friction on an horizontal plane (that also contains the circular path mentioned above).

22. A particle of mass m moves in a straight slot cut in a horizontal turntable that rotates at a constant angular speed of ω. The slot is at a distance R from the axis of rotation and symmetric about the diameter perpendicular to the slot. The particle is acted upon by a linear spring placed along the slot. The spring is unstretched when the particle is closest to the axis of rotation. Find the equation of motion of the particle.

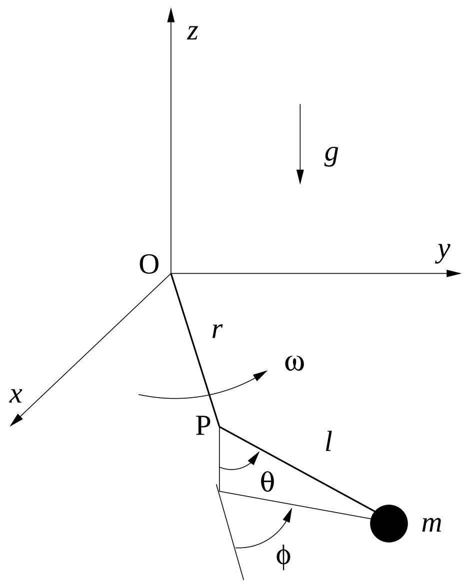
23. Recall that a cycloid is the curve traced by a point on the circumference of a circle that rotates without slipping on a straight line. Consider the case where the straight line is horizontal and the circle lies below the line. Show that a particle constrained to move along such a cycloid without friction and under uniform gravity is an *isochronous* pendulum, that is, a pendulum whose period is independent of the amplitude. This property is related to the fact (discovered by Huygens) that the cycloid is a *tautochrone*. (Hint: Use the arc length along the cycloid measured from the lowest point as the generalised coordinate. How is the arc length related to the angle of rotation of the circle tracing out the cycloid?)

24. A *governor* is a mechanical device that provides a speed feedback mechanism for regulating the speed of rotating machinery. The figure below shows a schematic of a governor. Two particles of mass m are located on arms that pivot at the top and the bottom. The ring mass M slides up and down on the shaft without friction. The equilibrium value θ_{eq} of the angle θ serves as a measurement of the speed ω of the shaft. Find the equation of the callibration curve (θ_{eq} as a function of ω) of the governor. (Hint: Find the equation of motion.)



25. A rod OP of length r rotates in the horizontal x-y plane at a constant rate ω. A pendulum of mass m and length l is attached to the end P of the rod. The orientation of the pendulum relative to the rod is given by the angle θ measured from the downward vertical, and the angle ϕ between the vertical plane through the pendulum and the direction of the rod as shown in the figure below. Using (θ, ϕ) as generalized coordinates,

write down Lagrange's equations for the system. Write down an integral of motion of the system.



26. A cart having mass M slides without friction on a horizontal track. The cart carries a frictionless hinge joint from which a particle of mass m is suspended using a massless rod of length l. Write down the equations of motion for this "pendulum on a cart". Include the effect of gravity.

Lagrange's Equations for Constrained Systems

27. A uniform rod of mass m and length l moves on the horizontal xy plane without friction. One end of the rod has a knife-edge constraint which prevents that end from having a velocity component perpendicular to the rod. Using the coordinates of the center of mass and the angular orientation as generalized coordinates, obtain the equations of motion of the rod.

28. A particle of mass m is acted upon by a conservative force field having the potential function $V = \frac{1}{2}(x^2 + y^2 + z^2)$ and a constraint described by $2\dot{x} + 3\dot{y} + 4\dot{z} + 5 = 0$. Find the equations of motion of the particle.

29. A particle of mass m slides on a smooth rigid wire having the shape $y = 3x^2$ with gravity acting in the negative y direction. Assuming the initial conditions $y(0) = y_0$, use the Lagrangian method to find the maximum constraint force during the resulting motion.

30. A particle of mass m slides without friction on a circular hoop under the action of gravity. The plane of the hoop is vertical and the hoop rotates about its vertical diameter with a constant angular velocity ω. Find the torque that needs to be applied to the hoop about the vertical axis in order to maintain its angular velocity.

31. A knife edge having mass m and moment of inertia J about the vertical axis slides without friction under gravity on a plane that is inclined at an angle α from the horizontal. What is the configuration space of the knife edge? Choose suitable generalised coordinates and use these to write down any contraints that may be present. If there are velocity constraints, are they integrable (that is, arise from position constraints) or not? Use the Lagrangian method to write down the equations of motion of the knife edge.

32. Two particles of mass m are connected by an inextensible string of length l. One particle slides without friction on a horizontal table. The other particle hangs from the string through a hole in the table, and is constrained to move only vertically. What is the configuration space of this system? Use the method of Lagrange multipliers to find an expression for the tension in the string as well as equations of motion of the first particle in polar coordinates.

Gibbs-Appell Equations

33. Obtain Gibbs-Appell equations for the system described in Problem 27.

34. Obtain Gibbs-Appell equations for the system described in Problem 28.

35. Obtain Gibbs-Appell equations for the system described in Problem 4.

36. A circular disc of mass m has moment of inertia I about the vertical axis passing through its center of mass. The disc is constrained to move and rotate only in the horizontal plane.

    (a) If x, y and ϕ denote the Cartesian coordinates of the center of mass and the orientation of the disc, respectively, then show that the Gibbs-Appell function for the disc is $\frac{1}{2}m(\ddot{x}^2 + \ddot{y}^2) + \frac{1}{2}I\ddot{\phi}^2 + \frac{1}{2}I\dot{\phi}^4$.

    (b) A knife edge at the center of mass of the disc constrains the velocity of the center of mass to lie along the knife edge. Use Gibbs-Appell equations to write down the complete set of equations of motion of the disc in terms of the speed v of the center of mass and the angular rate $\omega = \dot{\phi}$ about the vertical axis. Use the equations to show that the center of mass will either describe straight lines or circles.

    (c) Write down Lagrange's equations for the disc by using x, y and ϕ as generalized coordinates and eliminating constraint forces.

37. Two particles A and B of mass m are joined by a rigid, massless rod of length l. Particle A is subject to a knife-edge constraint that constrains the velocity of A to remain orthogonal to the rod at all times. The system of two particles thus formed moves in a horizontal two-dimensional plane. Write down Gibbs-Appell equations for the system using the Cartesian coordinates of A and the orientation of the rod AB as generalized coordinates.

Calculus of Variations

38. Show that if a curve renders the functional $I[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), \ddot{q}(t), t)dt$ stationary with respect to varied curves satisfying fixed boundary conditions on q and $\dot{q}$, then the curve satisfies
$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0.$$

39. Find the equation of a curve in the $x-y$ plane such that the area of the surface generated by rotating the curve about the y axis is a minimum from among all curves satisfying the same boundary conditions.

40. The refractive index n at a point in a certain liquid is a function of the depth y. As a result, any path traversed by light remains in a vertical plane (can you show this?). Fermat's principle states that the path followed by light between two locations (say $x = 0$ and $x = x_1$) in the vertical plane minimizes the time taken between the two

points. Use this principle to obtain an equation describing the path followed by light in the vertical $x - y$ plane.

41. The displacement of a simple harmonic oscillator is periodic and hence given by a Fourier series of the form $q(t) = \sum_{n=1}^{\infty} \cos n\omega t$ for appropriate initial conditions. Compute the action integral over the interval $[0, 2\pi/\omega]$ with the Lagrangian $L = m\dot{q}^2/2 - kq^2/2$. Show that the the action integral is an extremum for a nontrivial q only if $a_n = 0$ for every $n > 1$ and $\omega^2 = k/m$.

42. Geodesics on a surface are curves that render arclength (considered as a functional of curves defined on the surface) stationary. Show that every geodesic on a right circular cylinder is either a circle, a helix, or a straight line. Show that geodesics on a sphere are great circles.

43. Consider a circular cone having semi-vertex angle α and whose axis is vertical. Show that geodesics on the cone satisfy

$$r\frac{d^2r}{d\theta^2}\text{cosec}^2\alpha - 2\left(\frac{dr}{d\theta}\right)^2 \text{cosec}^2\alpha - r^2 = 0,$$

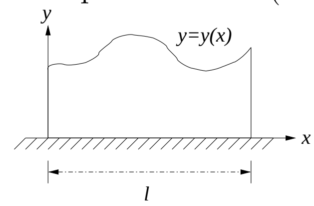
where r and θ are the polar coordinates of the horizontal projection of a point on the cone. Recall that geodesics are curves that render the arclength functional stationary.

44. Obtain equations describing a curve $x(t) = (q(t), p(t))$ that renders the functional $I[x] = \int_{t0}^{t1} F(q(\tau), \dot{q}(\tau), p(\tau), \tau)d\tau$ stationary among all curves defined on $[t_0, t_1]$ and satisfying $q(t_0) = q_{\mathrm{i}}$ and $q(t_1) = q_{\mathrm{f}}$. What do these equations yield when the function F is of the form $F(q, \dot{q}, p, t) = \mathrm{p}^{\mathrm{T}}\dot{q} - H(q, p, t)$?

45. The kinematic equations describing the motion of a particle moving in a two-dimensional plane with an acceleration that is constant in magnitude but varying in direction are given by $\dot{v} = m\cos\theta$, $\dot{\phi} = (m/v)\sin\theta$, where v is the speed (magnitude of velocity), m is the constant magnitude of the acceleration, heading angle ϕ is the angle between the velocity vector and a fixed reference axis in the plane, and θ is the angle between the acceleration vector and the velocity vector. This problem concerns the trajectory of **shortest arc length** that achieves a specified heading change from among all trajectories that achieve the same heading change. Show that the motion of the particle along such a **length-optimal** trajectory is such that $v^2 \sin\theta$ remains constant. Use this fact to show that the angle bisector between the acceleration and velocity vectors makes a constant angle with the reference axis. (Hint: Use the heading angle as the independent variable.)

46. The kinematic equations describing the motion of a particle moving in a two-dimensional plane with an acceleration that is constant in magnitude but varying in direction are given by $\dot{v} = m\cos\theta$, $\dot{\phi} = (m/v)\sin\theta$, where v is the speed (magnitude of velocity), m is the constant magnitude of the acceleration, heading angle ϕ is the angle between the velocity vector and a fixed reference axis in the plane, and θ is the angle between the acceleration vector and the velocity vector. This problem concerns the trajectory that achieves a specified heading change in the least time from among all trajectories that achieve the same heading change. Show that the motion of the particle along such a time-optimal trajectory is such that the velocity component perpendicular to the acceleration vector remains constant. Use this fact to show that the acceleration vector makes a constant angle with the reference axis. (Hint: Use the heading angle as the independent variable.)

47. The figure below shows the planform of a cantilever slab supported on one of its straight sides, which has a given length l. The slab is required to have a specified area A_1. For structural reasons, it is desired that the centroid of the slab should be as close to the support line as possible. Use calculus of variations to find the shape of the slab (that is, find the curve $y = y(x)$).



48. A simple closed planar curve is to be such that average distance of its points from the origin is R. Show that if the curve is to enclose minimum area, then it must be a circle of radius R centered at the origin. (Hint: Use polar coordinates.)

Conserved Quantities, Routhian Reduction and Noether's Theorem

49. A particle of mass m slides without friction on the surface of a circular cone under the action of gravity. The cone opens upwards, has its axis vertical and has an angle 2α at its apex. Write down the Lagrangian and the Jacobi integral using the distance r from the apex and the angular location ϕ about the axis of symmetry as generalized coordinates. Use the Routhian method to eliminate any ignorable coordinates. What is the Jacobi integral for the reduced system?

50. Two particles of mass m are connected by a rigid massless rod of length l. The dumbbell thus formed moves in a plane. Each particle is attracted towards the origin under the influence of a force that is proportional to the inverse square of its distance to the origin.

    (a) The Lagrangian of the system is invariant under the action of a group of transformations on the configuration space. Which group? Is it a one-parameter group?

    (b) Using the polar coordinates of one of the particles and the inclination of the dumbbell as generalized coordinates, obtain the Lagrangian of the system.

    (c) Describe the action of the transformation group you picked in a) above in terms of the generalized coordinates chosen in b). Verify that the Lagrangian is invariant under the action of the group.

    (d) Use Noether's theorem to obtain an integral of motion of the system.

    (e) What physical quantity do you think the integral of motion represents?

51. A particle is constrained to slide without friction under uniform gravity on the surface of revolution obtained by rotating the curve $z = x^2, y = 0$, about the z axis. Write down the Lagrangian for the particle using cylindrical coordinates, and identify ignorable coordinates. Apply the Routhian procedure to obtain a reduced set of equations for the non-ignorable cooordinates.

52. Two particles of mass m are connected by spring of unstretched length l, axial stiffness k, and negligible bending stiffness. One particle slides without friction on a horizontal table. The other particle hangs from the spring through a hole in the table and is constrained to move only vertically. Obtain the Lagrangian of the system using polar coordinates of the first particle and the vertical location of the second as generalised coordinates. Identify ignorable coordinates and use the Routhian method to obtain a reduced system of equations for the non-ignorable coordinates. Use the reduced system to suggest initial conditions for which the first particle performs circular motion.

53. A block of mass M with an inclined surface of inclination α slides without friction on a horizontal floor. A second block of mass m slides under gravity without friction on the inclined surface, and is attached to the top of the inclined surface with a spring of stiffness k and unstretched length l. Write down as many conserved quantities for the system as you can. Write down Hamilton's equations for the system. Use the horizontal location q_1 of M and the location q_2 of m along the inclined surface as generalized coordinates.

54. A dumbbell consists of two particles of mass m connected by a massless rod of length l. The dumbbell moves without friction on a horizontal plane. A knife-edge constraint at one (and only one) of the particles restricts the velocity of the particle to remain parallel to the connecting rod. List the ignorable coordinates of the system and corresponding conserved quantities if any. Is this system conservative? If so, then find the Jacobi integral and show that it is an integral of motion.

55. Two particles of mass m are connected by an inextensible string of length l. One particle slides without friction on a horizontal table. The other particle hangs from the string through a hole in the table and moves freely under gravity. Obtain the Lagrangian of the system using polar coordinates of the first particle and spherical coordinates of the second as generalised coordinates. Write down as many conserved quantities for this system as you can with justification. Identify ignorable coordinates and use the Routhian method to obtain a reduced system of equations for the non-ignorable coordinates.

Hamilton's Equations

56. Derive Hamilton's canonical equations for the system described in the previous question.

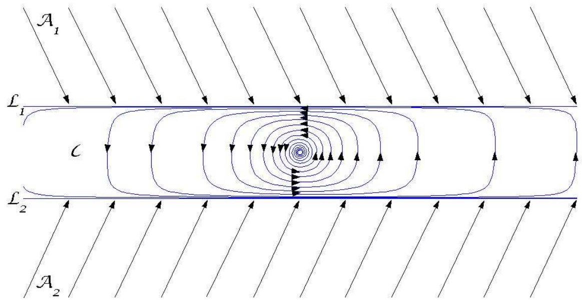
57. Derive Hamilton's canonical equations for a spherical pendulum consisting of a particle of mass m suspended from a point support by a rigid massless rod of length l.

58. A cylinder rotates freely about its axis of symmetry which is vertical. A particle of mass m slides freely under gravity on a helical track that is rigidly fixed to the outer surface of the cylinder. Obtain Hamilton's equations for the system using cylindrical coordinates of the particle as generalized coordinates.

59. A particle of mass m moves under the action of gravity inside a smooth circular tube whose plane remains vertical. The tube is free to rotate about a vertical axis passing through its center, and has inertia I about the axis of rotation and radius r. Write down Hamilton's equations for the system using suitable independent generalized coordinates.

60. Write down Hamilton's canonical equations for the system described in Problem 51 above.

61. The figures below show the trajectories of two systems in a two-dimensional phase space. Which of the two systems is likely to be described by Hamilton's canonical equations, and which is not? Explain briefly.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