04d01010 Classical Dynamics
15/04/08

Prob. 27] $V = \frac{1}{2}(x^2+y^2+z^2)$

Constraint $2\dot{x} + 3\dot{y} + 4\dot{z} + 5 = 0$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + 5 = 0 \quad \text{—(I)}$$

Now, $X = q = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Rightarrow \frac{\partial X}{\partial q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also, $F = -\frac{\partial V}{\partial X} = -\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Now, there are 3 unknowns and 1 constraint,
∴ the degree of freedom is 2.
∴ we need 2 linearly independent velocities that satisfy the constraint equation I.

∴ $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ satisfy the above condition.

$$\therefore \quad W = \begin{bmatrix} -1 & 2 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad \dot{X} = \dot{q} = Wv \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{— II}$$

Now $T' = W^T \left(\frac{\partial X}{\partial q}\right)^T F = \underset{2\times 3}{\begin{bmatrix} -1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}}$

$$\therefore \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix} = \begin{bmatrix} x+y \\ -2x+y-z \end{bmatrix}$$

$$\therefore \quad T_1 = x + y \qquad \text{—— (III)}$$
$$T_2 = -2x + y - z$$

Now, $S = \frac{1}{2} \ddot{x}^T J \ddot{x}$

$$\Rightarrow S = \frac{1}{2} m \left[\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2 \right]$$

from II, $\ddot{X} = \begin{bmatrix} -1 & 2 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix}$

$$\ddot{x} = -\dot{v}_1 + 2\dot{v}_2$$
$$\ddot{y} = -\dot{v}_1 - \dot{v}_2$$
$$\ddot{z} = \dot{v}_2$$

$$\therefore S = \frac{1}{2} m \left[(2\dot{v}_2 - \dot{v}_1)^2 + (-\dot{v}_1 - \dot{v}_2)^2 + \dot{v}_2^2 \right]$$

$$\frac{\partial S}{\partial \dot{v}} = \begin{bmatrix} \frac{\partial S}{\partial \dot{v}_1} \\ \frac{\partial S}{\partial \dot{v}_2} \end{bmatrix} = \frac{m}{2} \begin{bmatrix} -4\dot{v}_2 + 4\dot{v}_1 - 2\dot{v}_2 \\ -4\dot{v}_2 - 2\dot{v}_1 + 8\dot{v}_2 + 2\dot{v}_2 + 2\dot{v}_2 \end{bmatrix} = \frac{m}{2} \begin{bmatrix} -6\dot{v}_2 + 4\dot{v}_1 \\ -6\dot{v}_1 + 12\dot{v}_2 \end{bmatrix} \text{—— (IV)}$$

Now

$$\frac{\partial S}{\partial \dot{v}} = T'$$

$\therefore$ from 3 & 4.

$$m\left[-3\dot{v}_2 + 2\dot{v}_1\right] = x + y \Rightarrow \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = m \begin{bmatrix} -3 \\ 2 \end{bmatrix} \dot{v}$$

$$m\left[-3\dot{v}_1 + 6\dot{v}_2\right] = -2x + y - z \Rightarrow \begin{bmatrix} -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = m \begin{bmatrix} -3 \\ -6 \end{bmatrix} \dot{v}$$

These are the Gibbs Appel equations for the system.

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Classical Dynamics

33) $q(t) = \sum_{n=1}^{\infty} \cos n\omega t$

Action Integral $I = \int L\,dt$

$L = \frac{m\dot{q}^2}{2} - \cancel{mq}\ \frac{kq^2}{2}$

$\dot{q}(t) = -\omega \sum_{n=1}^{\infty} n \sin n\omega t$

$\therefore I = \int_0^{\frac{2\pi}{\omega}} \frac{m\omega^2}{2} \Big(\underbrace{\sum_{n=1}^{\infty} n \sin n\omega t}_{A}\Big)^2 dt - \frac{k}{2} \int_0^{\frac{2\pi}{\omega}} \Big(\underbrace{\sum_{n=1}^{\infty} \cos n\omega t}_{B_1}\Big)^2 dt$

$A = \sum_{n=1}^{\infty} n \sin n\omega t = \sin\omega t + 2\sin 2\omega t + 3\sin 3\omega t + \cdots$

$= (\sin\omega t + \sin 2\omega t + \sin 3\omega t + \cdots) + (\sin 2\omega t + \sin 3\omega t + \cdots) + (\sin 3\omega t + \cdots) + \cdots$

$= \left[\frac{e^{i\omega t} - e^{-i\omega t}}{2i} + \frac{e^{2i\omega t} - e^{-2i\omega t}}{2i} + \cdots\right] + \left[\frac{e^{2i\omega t} - e^{-2i\omega t}}{2i} + \frac{e^{3i\omega t} - e^{-3i\omega t}}{2i} + \cdots\right] + \cdots$

$= \left[\frac{(e^{i\omega t} + e^{2i\omega t} + \cdots)}{2i} - \frac{(e^{-i\omega t} + e^{-2i\omega t} + \cdots)}{2i}\right] + \left[\frac{(e^{2i\omega t} + e^{3i\omega t} + \cdots)}{2i} - \frac{(e^{-2i\omega t} + e^{-3i\omega t} + \cdots)}{2i}\right] + \cdots$

$= \left[\frac{e^{i\omega t}}{2i(1-e^{i\omega t})} - \frac{e^{-i\omega t}}{2i(1-e^{-i\omega t})}\right] + \left[\frac{e^{2i\omega t}}{2i(1-e^{i\omega t})} - \frac{e^{-2i\omega t}}{2i(1-e^{-i\omega t})}\right] + \cdots$

$= \frac{e^{i\omega t} + e^{2i\omega t} + e^{3i\omega t} + \cdots}{2i(1-e^{i\omega t})} - \frac{e^{-i\omega t} + e^{-2i\omega t} + e^{-3i\omega t} + \cdots}{2i(1-e^{-i\omega t})}$

$$\therefore A_1 = \frac{e^{i\omega t}}{2i(1-e^{i\omega t})^2} - \frac{e^{-i\omega t}}{2i(1-e^{-i\omega t})^2}$$

$$= \frac{e^{i\omega t}(1+e^{-2i\omega t}-2e^{-i\omega t}) - e^{-i\omega t}(1+e^{2i\omega t}-2e^{i\omega t})}{2i(1-e^{i\omega t})^2(1-e^{-i\omega t})^2}$$

$$= \frac{\cancel{e^{i\omega t}} + \cancel{e^{-i\omega t}} - \cancel{2} - \cancel{e^{-i\omega t}} - \cancel{e^{i\omega t}} + \cancel{2}}{2i(1-e^{i\omega t})^2(1-e^{-i\omega t})^2}$$

$$= 0$$

$$\therefore A_1 = 0$$

$$B_1 = \sum_{n=1}^{\infty}\cos n\omega t = \cos\omega t + \cos 2\omega t + \cos 3\omega t + \dots$$

$$= \frac{e^{i\omega t}+e^{-i\omega t}}{2} + \frac{e^{2i\omega t}+e^{-2i\omega t}}{2} + \frac{e^{3i\omega t}+e^{-3i\omega t}}{2} + \dots$$

$$= \frac{e^{i\omega t}+e^{2i\omega t}+e^{3i\omega t}+\dots}{2} + \frac{e^{-i\omega t}+e^{-2i\omega t}+e^{-3i\omega t}+\dots}{2}$$

$$= \frac{e^{i\omega t}}{2(1-e^{i\omega t})} + \frac{e^{-i\omega t}}{2(1-e^{-i\omega t})}$$

$$\therefore = \frac{e^{i\omega t} - 1 + e^{-i\omega t} - 1}{2(1 - e^{-i\omega t} - e^{i\omega t} + 1)} = -\frac{1}{2}$$

$$\therefore B_1 = -\frac{1}{2}$$

$$\therefore I = \frac{k}{2}\int_0^{\frac{2\pi}{\omega}} \frac{1}{4}\,dt$$

$$= \frac{k}{8} \times t\Big|_0^{\frac{2\pi}{\omega}} = \frac{2\pi k}{8\omega} = \frac{\pi k}{4\omega}$$

$$\therefore \boxed{\text{Action Integral} = \frac{\pi k}{4\omega}}$$

$$\text{Action Integral} = \int\left(\frac{m\dot{q}^2}{2} - \frac{kq}{2}\right)dt$$

For the action integral to be an extremum,

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$ $\cancel{= 0}$ has to be zero

$$L = \frac{m\dot{q}^2}{2} - \frac{kq}{2} \; ; \; \frac{\partial L}{\partial q} = -kq \; ; \; \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = m\frac{d}{dt}(\dot{q}) + kq = 0$$

$$\therefore -m\omega^2 \sum_{n=1}^{\infty} n^2 \cos n\omega t + k\sum_{n=1}^{\infty} \cos n\omega t = 0$$

$$\therefore \cos\omega t(-m\omega^2 + k) + \cos 2\omega t(-4m\omega^2 + k) + \dots = 0$$

$$\therefore \sum_{n=1}^{\infty} (k - m\omega^2 n^2)\cos n\omega t = 0$$

Assignment.

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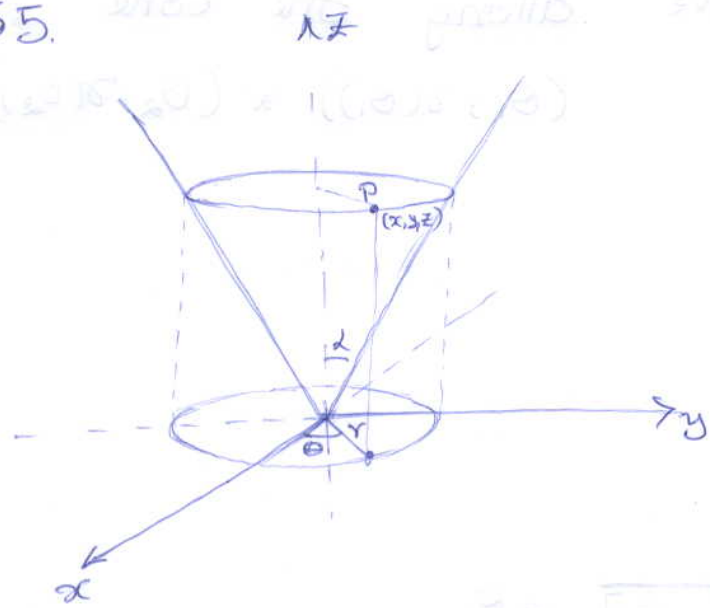
35.



$x = r\cos\theta$

$y = r\sin\theta$

$z = r\cot\alpha$.

Let ds be the infinetismal arc length on the cone then.

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$\therefore\ dx = -r\sin\theta\, d\theta + \cos\theta\, dr$$

$$dy = r\cos\theta\, d\theta + \sin\theta\, dr$$

$$dz = \cot\alpha\, dr.$$

$$(ds)^2 = r^2\sin^2\theta\,(d\theta)^2 + \cos^2\theta\,(dr)^2 - 2r\sin\theta\cos\theta\, dr\,d\theta$$
$$+ r^2\cos^2\theta\,(d\theta)^2 + \sin^2\theta\,(dr)^2 + 2r\cos\theta\sin\theta\, dr\,d\theta$$
$$+ \cot^2\alpha\,(dr)^2$$

$$(ds)^2 = \{r^2 + \dot{r}^2\,[1+\cot^2\alpha]\}\,(d\theta)^2$$

length of the curve allong the cone between the points $(\theta_1, r(\theta_1))$ & $(\theta_2, r(\theta_2))$ is then given by,

$$I = \int_{\theta_1}^{\theta_2} ds$$

$$I = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \dot{r}^2 [1 + \cot^2\alpha]}\, d\theta$$

I has an extremum when

$$L = \sqrt{r^2 + \dot{r}^2 [1 + \cot^2\alpha]} \quad \text{satisfies}$$
$$L = \sqrt{r^2 + \dot{r}^2 \operatorname{cosec}^2\alpha}.$$

$$\frac{d}{d\theta}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0.$$

$$\cancel{\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}\left(r^2 + \dot{r}^2 [\operatorname{cosec}^2\alpha]\right)^{-1/2}}$$

Let $r^2 + \dot{r}^2 \operatorname{cosec}^2\alpha = P \Rightarrow \frac{\partial P}{\partial \dot{r}} = 2\dot{r}\operatorname{cosec}^2\alpha$

then $L = P^{1/2}$

$$\frac{d}{d\theta}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{d\theta}\left(\frac{1}{2} P^{-1/2} \frac{\partial P}{\partial \dot{r}}\right) - \frac{1}{2} P^{-1/2} \frac{\partial P}{\partial r} = 0$$

i.e.

$$\frac{1}{2}\left(-\frac{1}{2}\right) P^{-3/2} \frac{dP}{d\theta} \frac{\partial P}{\partial \dot{r}} + \frac{1}{2} P^{-1/2} \frac{d}{d\theta}\left(\frac{\partial P}{\partial \dot{r}}\right) - \frac{1}{2} P^{-1/2} \frac{\partial P}{\partial r} = 0$$

$$\left\{-\frac{1}{4}\left(r^2+\dot{r}^2\operatorname{cosec}^2\alpha\right)^{-3/2}\left(2r\dot{r}+2\dot{r}\ddot{r}\operatorname{cosec}^2\alpha\right)2\dot{r}\operatorname{cosec}^2\alpha\right\}$$
$$+\left\{\frac{1}{2}\left(r^2+\dot{r}^2\operatorname{cosec}^2\alpha\right)^{-1/2}\ddot{r}\operatorname{cosec}^2\alpha\right\} - \left\{\frac{1}{2}\left(r^2+\dot{r}^2\operatorname{cosec}^2\alpha\right)^{-1/2} 2r\right\}$$
$$= 0.$$

$$-\left(r+\ddot{r}\operatorname{cosec}^2\alpha\right)\dot{r}^2\operatorname{cosec}^2\alpha$$
$$+\left(r^2+\dot{r}^2\operatorname{cosec}^2\alpha\right)\ddot{r}\operatorname{cosec}^2\alpha$$
$$-\left(r^2+\dot{r}^2\operatorname{cosec}^2\alpha\right)r = 0.$$

$$-r\dot{r}^2\operatorname{cosec}^2\alpha - \cancel{\ddot{r}\dot{r}^2\operatorname{cosec}^4\alpha} + r^2\ddot{r}\operatorname{cosec}^2\alpha$$
$$+\cancel{\dot{r}^2\ddot{r}\operatorname{cosec}^4\alpha} - r^3 - r\dot{r}^2\operatorname{cosec}^2\alpha = 0$$

$$r\ddot{r}\operatorname{cosec}^2\alpha - 2\dot{r}^2\operatorname{cosec}^2\alpha - r^2 = 0$$

$$\boxed{r\left(\frac{d^2r}{d\theta^2}\right)\operatorname{cosec}^2\alpha - 2\left(\frac{dr}{d\theta}\right)^2\operatorname{cosec}^2\alpha - r^2 = 0}$$

Given, $I(P,q) = \int_{t_0}^{t_f} F(q,\dot{q},p,t)\,dt$

we have to find $x(t) = (p(t), q(t))$ that renders the integral stationary.

Let, $x^*(t) = (p^*(t), q^*(t))$ be the curve which makes I stationary ...... (1)

consider, perturbations $h_1(t)$ and $h_2(t)$ in $q^*(t)$ and $p^*(t)$ respectively. $t \in [t_0, t_f]$

$$\left.\begin{aligned} \therefore\quad q(t) &= q^*(t) + s_1 h_1(t) \\ p(t) &= p^*(t) + s_2 h_2(t) \end{aligned}\right\} \text{ $(p(t), q(t))$ represent ~~one~~ family of curves.}$$

$$\left.\begin{aligned} q(t_f) &= q^*(t_f) = q_f \\ q(t_0) &= q^*(t_0) = q_i \end{aligned}\right\} \text{ ~~given~~ given, fixed endpoints of $q(t)$}$$

$$\left.\begin{aligned} \therefore\quad h_1(t_0) &= 0 \\ \text{and}\quad h_1(t_f) &= 0 \end{aligned}\right\} \ldots\ldots (2)$$

For (1) to hold true, $\frac{\partial I}{\partial s_1} = 0$ and $\frac{\partial I}{\partial s_2} = 0$

$$I(P,q) = \int_{t_0}^{t_f} F(q,\dot{q},p,t)\,d\tau$$

$$\frac{\partial I}{\partial s_1} = \int_{t_0}^{t_f} \frac{\partial F(q,\dot{q},P,t)}{\partial s_1}\,d\tau$$

$$= \int_{t_0}^{t_f} \left(\frac{\partial F}{\partial q}\frac{\partial q}{\partial s_1} + \frac{\partial F}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial s_1} + \frac{\partial F}{\partial P}\cdot\underbrace{\frac{\partial P}{\partial s_1}}_{=0}\right) d\tau$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial q} h_1(\tau) + \frac{\partial F}{\partial \dot{q}} \dot{h}_1(\tau) + 0\right) d\tau$$

$$= \int_{t_0}^{t_1} \frac{\partial F}{\partial q} h_1(\tau)\,d\tau + \left[\frac{\partial F}{\partial \dot{q}} h_1(\tau)\right]_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}}\right) h_1(\tau)\,d\tau$$

(Integration by parts)

$$= \int_{t_0}^{t_f} \left[\frac{\partial F}{\partial q} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot q}\right)\right] h_1(\tau)\, d\tau$$

$$= 0$$

for any arbitrary $h_1(\tau)$, above ~~unused~~ holds true,

$$\therefore \frac{\partial F}{\partial q} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot q}\right) = 0 \qquad (I)$$

$$\frac{\partial I}{\partial s_2} = \int_{t_0}^{t_f} \frac{\partial F(q, \dot q, p, t)}{\partial s_2}\, d\tau$$

$$= \int_{t_0}^{t_f} \left(\frac{\partial F}{\partial q}\underbrace{\frac{\partial q}{\partial s_2}}_{=0} + \frac{\partial F}{\partial \dot q}\underbrace{\frac{\partial \dot q}{\partial s_2}}_{=0} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial s_2}\right) d\tau$$

$$= \int_{t_0}^{t_f} \frac{\partial F}{\partial p} \cdot h_2(\tau)\, d\tau$$

This holds true for any arbitrary $h_2(\tau)$

$$\therefore \frac{\partial F}{\partial p} = 0 \cdots\cdots \qquad (II)$$

(I) and (II) are the equations of the desired curve $x(t)$, which makes I stationary.

consider, $F = p^T \dot q - H(p, q, t)$

$$(I) \Rightarrow \frac{\partial}{\partial q}\left[p^T \dot q - H(p, q, t)\right] - \frac{d}{dt}(p^T) = 0$$

$$\therefore -\frac{\partial H}{\partial q} - \frac{d(p^T)}{dt} = 0$$

$$\therefore (\dot p)^T = -\frac{\partial H}{\partial q}$$

$$\therefore \dot p = -\left(\frac{\partial H}{\partial q}\right)^T$$

(II) $$\frac{\partial F}{\partial p} = 0$$

$$\therefore \frac{\partial}{\partial p}(p^T \dot{q}) - \frac{\partial H}{\partial p} = 0$$

$$\therefore \dot{q} - \frac{\partial H}{\partial p} = 0$$

$$\therefore \dot{q} = \frac{\partial H}{\partial p}.$$

$$\dot{p} = -\left(\frac{\partial H}{\partial q}\right)^T, \quad \dot{q} = \frac{\partial H}{\partial p}.$$

are infact Hamiltonian equations of dynamics.

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38). Given $\dot{v} = m \cos\theta$

$$\dot{\phi} = \frac{m}{V}\sin\theta$$

Using these we write

$$\frac{\partial v}{\partial \phi} = \frac{v}{\tan\theta} = v' \text{ (say)}. \quad --- \quad (1)$$

To minimize time taken for given heading change.

$$\text{Time taken} = \int_{\phi_1}^{\phi_2} \frac{d\phi}{\dot{\phi}}$$

$$= \int \frac{v}{m\sin\theta} \cdot d\phi$$

$$= \int \frac{\sqrt{v^2+v'^2}}{m}\, d\phi \qquad \left[\text{From -- (1) we say } \sin\theta = \frac{v}{\sqrt{v^2+v'^2}}\right]$$

As acceleration is constant, we can consider

$$L(v, v', \phi) = \frac{\sqrt{v^2+v'^2}}{m}$$

Now,

$$\frac{\partial L}{\partial v'} = \frac{v'}{m\sqrt{v^2+v'^2}} \quad \& \quad \frac{\partial L}{\partial v} = \frac{v}{m\sqrt{v^2+v'^2}}$$

$$\cancel{\frac{\partial}{\partial\phi}} \quad \frac{d}{d\phi}\left(\frac{\partial L}{\partial v'}\right) = \frac{v''}{m\sqrt{v^2+v'^2}} - \frac{v'(vv' + v'v'')}{m(v^2+v'^2)^{3/2}}$$

$$= \frac{v''v^2 - vv'^2}{m(v^2+v'^2)^{3/2}}$$

Now, using Lagrange Equ.

$$\frac{d}{d\phi}\left(\frac{\partial L}{\partial v'}\right) - \frac{\partial L}{\partial v} = 0$$

we get,

$$\frac{v'' v^2 - 2vv'^2 - v^3}{m(v^2 + v'^2)^{3/2}} = 0 \quad \text{---} \quad (2)$$

Now consider

$$\frac{\partial}{\partial \phi}(v \sin\theta) = \frac{\partial}{\partial \phi}\left(\frac{v^2}{\sqrt{v^2 + v'^2}}\right)$$

$$= \frac{2vv'(v^2 + v'^2) - v^2(vv' + v'v'')}{(v^2 + v'^2)^{3/2}}$$

$$= \frac{v'(v^3 + 2v'^2 . v - v^2 v'')}{(v^2 + v'^2)^{3/2}}$$

Putting this result in equ. (2), we get

$$\frac{-1}{mv'} \frac{\partial}{\partial \phi}(v \sin\theta) = 0$$

Thus $v \sin\theta$ is constant. As θ is angle betn acc. & velocity vector. Hence $v.\sin\theta$ is component of velocity perpendicular to accelaration vector. And is constant.

Classical Dynamics

40)

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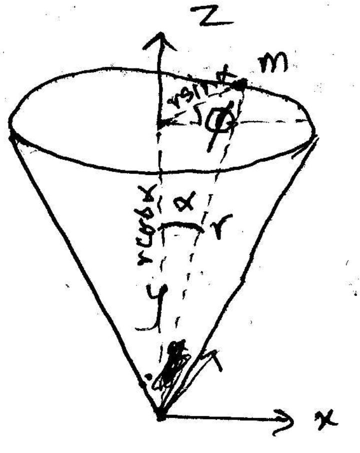


Generalized coordinates

$\Rightarrow r, \phi$

Cartesian coordinates of the particle are,

$x = r\sin\alpha\cos\phi \quad , \quad \dot{x} = \sin\alpha\,[\dot{r}\cos\phi - r\sin\phi\,\dot{\phi}]$

$y = r\sin\alpha\sin\phi \quad , \quad \dot{y} = \sin\alpha\,[\dot{r}\sin\phi + r\cos\phi\,\dot{\phi}]$

$z = r\cos\alpha \quad , \quad \dot{z} = \cos\alpha\,[\dot{r}]$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$= \frac{1}{2} m [\sin^2\alpha\,(\dot{r}^2 + r^2\dot{\phi}^2) + \dot{r}^2\cos^2\alpha]$

$= \frac{1}{2} m [\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha]$

$V = mgz = mgr\cos\alpha$

$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha) - mgr\cos\alpha$

L is independent of ϕ, $\therefore$ ϕ is an ignorable coordinate

$P_\phi = \frac{\partial L}{\partial \dot{\phi}}$ is conserved

$\therefore \quad P_\phi = mr^2\dot{\phi}\sin^2\alpha \Rightarrow \dot{\phi} = \frac{P_\phi}{mr^2\sin^2\alpha}$

Jacobi Integral,

$$h = \frac{\partial L}{\partial \dot\phi}\dot\phi - L = mr^2\dot\phi^2\sin^2\alpha - \frac{1}{2}m\left[\dot r^2 + r^2\dot\phi^2\sin^2\alpha\right] + mgr\cos\alpha$$

$$\therefore h = \frac{1}{2}m\left[r^2\dot\phi^2\sin^2\alpha - \dot r^2\right] + mgr\cos\alpha$$

Routhian Reduction →

$$R = L - p_\phi\dot\phi$$

$$= \frac{1}{2}m(\dot r^2 + r^2\dot\phi^2\sin^2\alpha) - mgr\cos\alpha - mr^2\dot\phi^2\sin^2\alpha$$

$$= \frac{1}{2}m(\dot r^2 - r^2\dot\phi^2\sin^2\alpha) - mgr\cos\alpha$$

$$= \frac{1}{2}m\dot r^2 - \frac{1}{2}\frac{p_\phi^2}{mr^2\sin^2\alpha} - mgr\cos\alpha$$

R satisfies the equation,

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot r}\right) - \frac{\partial R}{\partial r} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot r) - \left[\frac{p_\phi^2}{mr^3\sin^2\alpha} - mg\cos\alpha\right] = 0$$

$$\Rightarrow m\ddot r - \frac{p_\phi^2}{mr^3\sin^2\alpha} - mg\cos\alpha = 0 \qquad -(1)$$ (independent of ϕ)

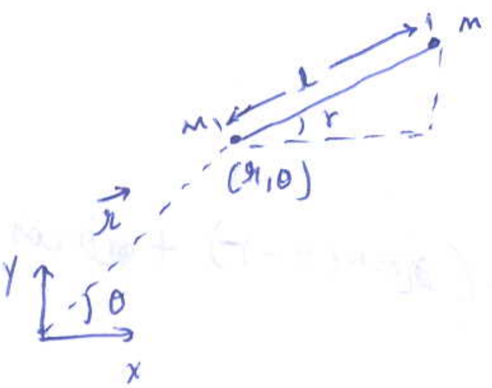
Also, $$\dot\phi = -\frac{\partial R}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\alpha} \qquad -(2)$$

Jacobi integral of the reduced system,

$$h = \frac{1}{2}m\left[r^2\frac{p_\phi^2}{(mr^2\sin^2\alpha)^2} - \dot r^2\right] + mgr\cos\alpha$$

$$= \frac{1}{2}\frac{p_\phi^2}{mr^2\sin^4\alpha} - \frac{1}{2}m\dot r^2 + mgr\cos\alpha$$ (independent of ϕ)

41]



(a). Configuration space of the system is $\mathbb{R}^2 \times S^1$

Let (r,θ) be the polar co-ordinates of the 1st particle and γ be the inclination of the dumbell w.r.t. the x-axis.
Distance of the 2nd particle from the origin is given by

$$r_2 = \sqrt{(r\cos\theta + l\cos\gamma)^2 + (r\sin\theta + l\sin\gamma)^2}$$

$$\therefore r_2 = \sqrt{r^2 + l^2 + 2lr(\cos(\theta-\gamma))} \quad \dots (1).$$

The force ~~of~~ on the system varies as the inverse square law. Let the constant be μ. $\vec{F} = \frac{\mu}{r^2}\hat{r}$

$\therefore$ The Lagrangian is invariant under the group of transformations which leaves $(\theta-\gamma)$ the same.

viz. ~~if $q = \begin{bmatrix} r \\ \theta \\ \gamma \end{bmatrix}$ $h_s(q) =$~~

viz. if $q = \begin{bmatrix} r \\ \theta \\ \gamma \end{bmatrix}$ $h_s(q) = \begin{bmatrix} r \\ \theta+s \\ \gamma+s \end{bmatrix}$ $s \in \mathbb{R}$. (2)

Thus, it is a one parameter group.

(b). $V = -\mu\left[\frac{1}{r} + \frac{1}{r_2}\right]$

$$\therefore V = -\mu\left\{\frac{1}{r} + \frac{1}{(r^2+l^2+2lr\cos(\theta-\gamma))^{1/2}}\right\} \quad \dots (3)$$

~~$T = \frac{1}{2}$~~ Cartesian coordinates of the second mass are given by,

$$x_2 = r\cos\theta + l\cos\gamma$$
$$y_2 = r\sin\theta + l\sin\gamma$$

$$T = \frac{1}{2} m\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{x}_2^2 + \dot{y}_2^2\right]$$

$$\therefore T = \frac{1}{2} m\left[2(\dot{r}^2 + r^2\dot{\theta}^2) + l^2\dot{r}^2 + 2l\dot{r}\left(\dot{r}\sin(\theta-\gamma) + r\dot{\theta}\cos(\theta-\gamma)\right)\right] \quad (4)$$

$$L = T - V$$

$$\therefore L = \frac{m}{2}\left\{2(\dot{r}^2 + r^2\dot{\theta}^2) + l^2\dot{r}^2 + 2l\dot{r}\left(\dot{r}\sin(\theta-\gamma) + r\dot{\theta}\cos(\theta-\gamma)\right)\right\} \quad (5)$$

(c). As stated earlier (2) L is invariant under the one parameter group of transformations,

$$q = \begin{bmatrix} r \\ \theta \\ \gamma \end{bmatrix}, \quad h_s(q) = \begin{bmatrix} r \\ \theta + s \\ \gamma + s \end{bmatrix} \quad (6)$$

from (5), it can be seen that as r, $\dot{r}$, $\dot{\theta}$, and $(\theta-\gamma)$ remain unchanged, L remains the same

(d) from (6), $$\frac{d}{ds}\Big|_{s=0} h_s(q) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad (7)$$

$$\frac{\partial L}{\partial \dot{q}} = \left[\frac{\partial L}{\partial \dot{r}} \quad \frac{\partial L}{\partial \dot{\theta}} \quad \frac{\partial L}{\partial \dot{\gamma}}\right] \quad (8)$$

By Noether's theorem, an integral of motion is.

$$p(q,\dot{q},t) = \frac{\partial L}{\partial \dot{q}} \times \frac{d}{ds}\Big|_{s=0} h_s(q)$$

$$p(q,\dot{q},t) = \frac{\partial L}{\partial \dot{\theta}} + \frac{\partial L}{\partial \dot{\gamma}}$$

$\frac{\partial L}{\partial \dot{r}} = 0$ from (5)

$$\frac{\partial L}{\partial \dot{\theta}} = m\left[r^2 2\dot{\theta}\right]$$

$$\therefore \frac{\partial L}{\partial \dot{\theta}} = 2mr^2\dot{\theta} \quad \ldots (9).$$

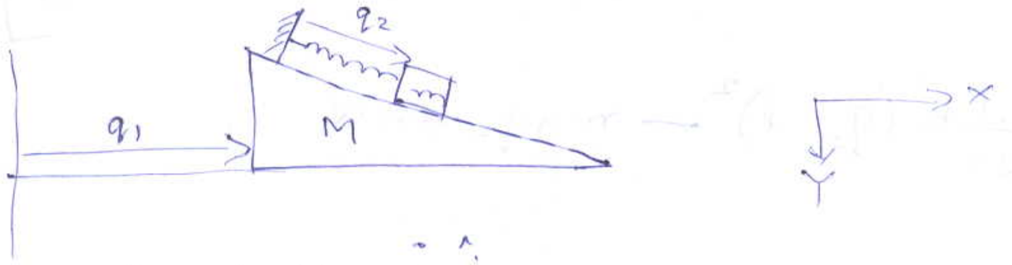
$\therefore$ Integral of motion is, $p(q, \dot{q}, t) = 2mr^2\dot{\theta}$ ..(10)

(e). The obtained integral of motion represents the angular momentum of the system about the origin which is conserved under the given one parameter transformations group.

CLASSICAL DYNAMICS Ashok Rajani

04103009

(43)



velocity of $M = \dot{q}_1 \hat{i}$

velocity of $m = (\dot{q}_1 + \dot{q}_2 \cos\alpha)\hat{i} + (\dot{q}_2 \sin\alpha)\hat{j}$

$$\therefore L = \frac{1}{2}\left[M(\dot{q}_1)^2 + m\left[(\dot{q}_1 + \dot{q}_2\cos\alpha)^2 + (\dot{q}_2 \sin\alpha)^2\right]\right]$$

$$- \frac{1}{2}k(q_2 - l)^2 + mgq_2 \sin\alpha$$

since L is independent of q_1, $\frac{\partial L}{\partial \dot{q}_1}$ is conserved

$$\therefore M\dot{q}_1 + m(\dot{q}_1 + \dot{q}_2 \cos\alpha) = 0$$

Also,

(i) All forces are conservative (Only gravity force).

(ii) No velocity constraints

(iii) Lagrangian is independent of time.

Hence, $h = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$ is conserved.

since, system is a natural system,

h is the total energy.

Hence, total energy is conserved.

$$\therefore \frac{1}{2}\left[M\dot{q}_1^2 + m\left\{(\dot{q}_1+\dot{q}_2\cos\alpha)^2+(\dot{q}_2\sin\alpha)^2\right\}\right]$$

$$+\frac{1}{2}k(q_2-l)^2 - mgq_2\sin\alpha$$

HAMILTON'S EQUATIONS:

$$p_{q_1} = \frac{\partial L}{\partial \dot{q}_1} = M\dot{q}_1 + m(\dot{q}_2\cos\alpha + \dot{q}_1)$$

$$p_{q_2} = \frac{\partial L}{\partial \dot{q}_2} = m\dot{q}_2\sin^2\alpha + m(\dot{q}_2\cos\alpha+\dot{q}_1)\cos\alpha$$

$$= m[\dot{q}_1\cos\alpha + \dot{q}_2]$$

$$\therefore \dot{q}_1 = \frac{p_{q_1} - p_{q_2}\cos\alpha}{M+m\sin^2\alpha}$$

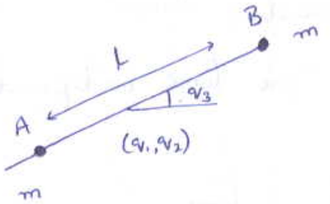
$$\dot{q}_2 = \frac{(M+m)p_{q_2} - (m\cos\alpha)p_{q_1}}{m(M+m\sin^2\alpha)}$$

$H = p_{q_1}\cdot\dot{q}_1 + p_{q_2}\cdot\dot{q}_2 - L$, $\dot{q}_1$ & $\dot{q}_2$ are substituted as above.

p_{q_1} is conserved $\Rightarrow$ $\dot{p}_{q_1} = 0$

$$\dot{p}_{q_2} = -\frac{\partial H}{\partial q_2} = -k(q_2-l) + mg\sin\alpha.$$

(44)



Given this system on a frictionless horizontal plane & velocity of A constrained to ~~move~~ be along the rod

Classical dynamics
Ch. L. Rama Rao
04001017.

First to find the ignorable co-ordinates, we need L

$L = T - V$

$V = 0$ as there are no conservative forces {except gravity which is already taken care of}

$$T = \frac{1}{2}(2m)(\dot q_1^2 + \dot q_2^2) + \frac{1}{2}\left(2\cdot m(l/2)^2\right)\dot q_3^2$$

{Translational K.E of COM + Rotational}

$$= m(\dot q_1^2 + \dot q_2^2) + \frac{ml^2}{4}\dot q_3^2$$

$$\therefore \quad L = m(\dot q_1^2 + \dot q_2^2) + \frac{ml^2}{4}\dot q_3^2$$

L is independent of q_1, q_2 & q_3. They are the ignorable coordinates

The corresponding conserved quantities (corresponding momenta)

$$P_{q_1} = \frac{\partial L}{\partial \dot q_1} = 2m\dot q_1 \qquad P_{q_3} = \frac{\partial L}{\partial \dot q_3} = \frac{ml^2}{2}\dot q_3$$

$$P_{q_2} = \frac{\partial L}{\partial \dot q_2} = 2m\dot q_2$$

* The velocity constraint is also time independent

$$\begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} q_1 - l/2\cos q_3 \\ q_2 - l/2\sin q_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot x_A \\ \dot y_A \end{bmatrix} = \begin{bmatrix} \dot q_1 + l/2\sin q_3\,\dot q_3 \\ \dot q_2 - l/2\cos q_3\,\dot q_3 \end{bmatrix}$$

velocity perpendicular to rod : $V_{A\perp} = 0$

$$\Rightarrow \quad \dot y_A\cos q_3 - \dot x_A\sin q_3 = 0 \quad \Rightarrow \quad \dot q_2\cos q_3 - \dot q_1\sin q_3 - l/2\,\dot q_3 = 0$$

of the standard form $a\dot q = 0$

Now, finding the Jacobi integral

$$h = \frac{\partial L}{\partial \dot q}\dot q - L = \begin{bmatrix} 2m\dot q_1 & 2m\dot q_2 & \frac{ml^2}{2}\dot q_3 \end{bmatrix}\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \end{bmatrix} - \left\{ m(\dot q_1^2 + \dot q_2^2) + \frac{ml^2}{4}\dot q_3^2 \right\}$$

$$= m\dot q_1^2 + m\dot q_2^2 + \frac{ml^2}{4}\dot q_3^2$$

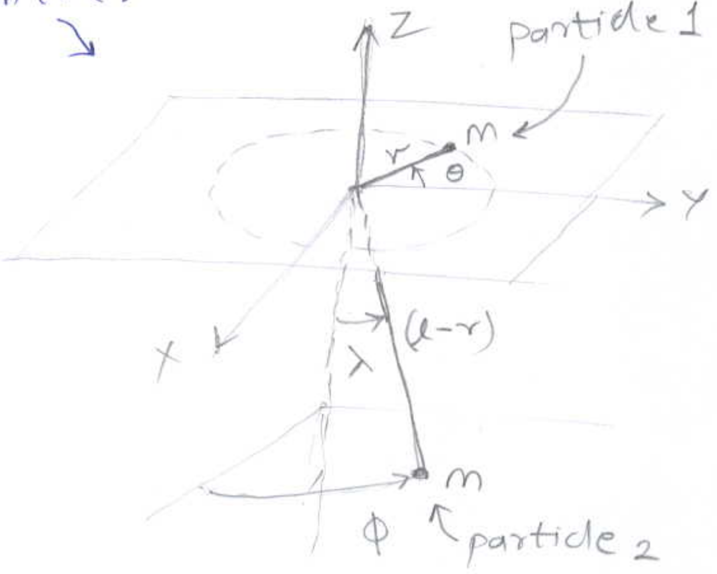
* We have a system, where there are no non conservative forces
We have already shown that velocity constraints are time independent & of the form $a\dot{q} = 0$
We also see that Lagrangian (L) is independent of time
∴ The Jacobi integral is a constant (ie) a first integral of motion.
* The other way of proving this would be to show
$\frac{d}{dt}(h) = 0$ by using equations of motion.

15/04/08 AE 459: Classical Dynamics

Submitted by
Mandar Kulkarni
(04d01009)

PROBLEM NO. 45

Particles 1 and 2 are shown here.



Gen. coordinates ~~q~~ are

$q = \{r, \theta, \lambda, \phi\}$

$\therefore L = \frac{1}{2} m \Big(\dot{r}^2 + r^2 \dot{\theta}^2 + (l-r)^2 \dot{\lambda}^2 + (l-r)^2 \dot{\phi}^2 \sin^2 \lambda \Big) - mg(l-r)(1-\cos\lambda)$

It is thus observed that $\frac{\partial L}{\partial \theta} = 0 = \frac{\partial L}{\partial \phi}$

$\Rightarrow$ $\boxed{\theta \;\&\; \phi \text{ are ignorable coordinates}}$

$\Rightarrow$ $\boxed{p_\theta \;\&\; p_\phi \text{ are conserved}}$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$, $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m (l-r)^2 \sin^2 \lambda \, \dot{\phi}$

Other conserved quantities:-

The system has

(i) Only conservative forces

(ii) No velocity constraints (for the chosen generalized coordinates)

(iii) $\frac{\partial L}{\partial t} = 0$, i.e. Lagrangian is independent of time variable

(iv) Also, there are no moving surfaces,

i.e. $\frac{\partial x}{\partial t} = 0$, where ~~$x = \{x_1, y_1, x_2, y_2\}^T$~~

$x = \{\underbrace{x_1, y_1, z_1}_{\text{cartesian coods. of particle 1}}, \underbrace{x_2, y_2, z_2}_{\text{cartesian coods. of particle 2}}\}^T$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\lambda}}\right) = m(l-r)^2 \ddot{\lambda}$$

$$\frac{\partial R}{\partial \lambda} = \frac{p_\phi^2 \cos\lambda}{m(l-r)^2 \sin^3\lambda} - mg(l-r)\sin\lambda.$$

$$\therefore \boxed{\ddot{\lambda} = \frac{p_\phi^2 \cos\lambda}{m^2 (l-r)^4 \sin^3\lambda} - \frac{g \sin\lambda}{(l-r)}}$$

p_θ and p_ϕ $\rightarrow$ are obtained from initial conditions.

Hence,

$$h = \frac{\partial L}{\partial \dot{q}}\dot{q} - L \rightarrow \text{is a conserved quantity}$$

and in this case, $h \rightarrow$ is the total energy of the system

$$\therefore \boxed{h = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + (l-r)^2\dot{\lambda}^2 + (l-r)^2\dot{\phi}^2\sin^2\lambda\right) + mg(l-r)(1-\cos\lambda)} \rightarrow \text{is conserved}$$

Thus the conserved quantities are: p_θ, p_ϕ, h.

Routhian,

$$R = L - p_I^T \dot{q}_I \; .$$

$$p_I^T \dot{q} = \begin{bmatrix} p_\theta \\ p_\phi \end{bmatrix}^T [\dot{\theta} \;\; \dot{\phi}]^T = \begin{bmatrix} mr^2\dot{\theta} \\ m(l-r)^2\dot{\phi}\sin^2\lambda \end{bmatrix}^T [\dot{\theta} \;\; \dot{\phi}]^T$$

$$= mr^2\dot{\theta}^2 + m(l-r)^2\dot{\phi}^2\sin^2\lambda \quad \ldots \text{ but } \dot{\theta} = \frac{p_\theta}{mr^2}, \; \dot{\phi} = \frac{p_\phi}{m(l-r)^2\sin^2\lambda}$$

$$= \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{m(l-r)^2\sin^2\lambda}$$

$$\therefore R = L - p_I^T\dot{q}$$

$$= \frac{1}{2}m\left(\dot{r}^2 + (l-r)^2\dot{\lambda}^2\right) - \frac{1}{2}\frac{p_\theta^2}{mr^2} - \frac{1}{2}\frac{p_\phi^2}{m(l-r)^2\sin^2\lambda} - mg(l-r)(1-\cos\lambda)$$

Equations for non-ignorable coordinates: r & λ :-

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) = m\ddot{r},$$

$$\frac{\partial R}{\partial r} = -m(l-r)\dot{\lambda}^2 + \frac{p_\theta^2}{mr^3} - \frac{p_\phi^2}{m(l-r)^3\sin^2\lambda} + mg(1-\cos\lambda)$$

$$\therefore \boxed{\ddot{r} = (l-r)^2\dot{\lambda}^2 + \frac{p_\theta^2}{m^2r^3} - \frac{p_\phi^2}{m^2(l-r)^3\sin^2\lambda} + g(1-\cos\lambda)}$$

Classical Dynamics by: 04D01016 Parikshit

Prob No 47 Spherical Pendulum

$$T = \frac{1}{2} m l^2 \left(\dot\theta^2 + \dot\phi^2 \sin^2\theta \right)$$

$$V = mgl\,(1-\cos\theta)$$

$$L = T - V = \frac{1}{2} m l^2 \left(\dot\theta^2 + \dot\phi^2 \sin^2\theta \right) - mgl\,(1-\cos\theta)$$

$$P_\theta = \frac{\partial L}{\partial \dot\theta} = m l^2 \dot\theta$$

$$P_\phi = \frac{\partial L}{\partial \dot\phi} = m l^2 \sin^2\theta\, \dot\phi$$

$$P = \begin{bmatrix} m l^2 & \\ & m l^2 \sin^2\theta \end{bmatrix} \begin{bmatrix} \dot\theta & \dot\phi \end{bmatrix}$$

$$H = p^T \dot q - L$$

$$= \begin{bmatrix} m l^2 \dot\theta & m l^2 \sin^2\theta\, \dot\phi \end{bmatrix} \begin{bmatrix} \dot\theta \\ \dot\phi \end{bmatrix} - L$$

$$= m l^2 \dot\theta^2 + m l^2 \sin^2\theta\, \dot\phi^2 - L$$

$$= P_\theta \dot\theta + P_\phi \dot\phi - \frac{1}{2} m l^2 \left(\dot\theta^2 + \dot\phi^2 \sin^2\theta \right) + mgl\,(1-\cos\theta)$$

$$H = \frac{1}{2} m l^2 \dot\theta^2 + \frac{1}{2} m l^2 \sin^2\theta\, \dot\phi^2 + mgl\,(1-\cos\theta)$$

$$H = \frac{1}{2} \frac{P_\theta^2}{m l^2} + \frac{1}{2} \frac{P_\phi^2}{m l^2 \sin\theta} + mgl\,(1-\cos\theta)$$

$$\frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{m l^2} \qquad \frac{\partial H}{\partial \phi} = \frac{P_\phi}{m l^2 \sin\theta}$$

$$\frac{\partial H}{\partial \theta} = m l^2 \sin\theta \cos\theta\, \dot\phi^2 + mgl \sin\theta$$

$$\frac{\partial H}{\partial \phi} = 0$$

Hamilton's Canonical Eqns.

$$\frac{\partial H}{\partial p} = \dot{q} \quad \& \quad \frac{\partial H}{\partial q} = -\dot{p}$$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{p_\theta}{ml^2} \qquad \frac{\partial H}{\partial p_\phi} = \dot{\phi} = \frac{p_\phi}{ml^2 \sin\theta}$$

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta = ml^2 \sin\theta \cos\theta \, \dot{\phi}^2 + mgl \sin\theta$$

$$\frac{\partial H}{\partial \phi} = \dot{p}_\phi = 0$$

$$\dot{\theta} = \frac{p_\theta}{ml^2}$$

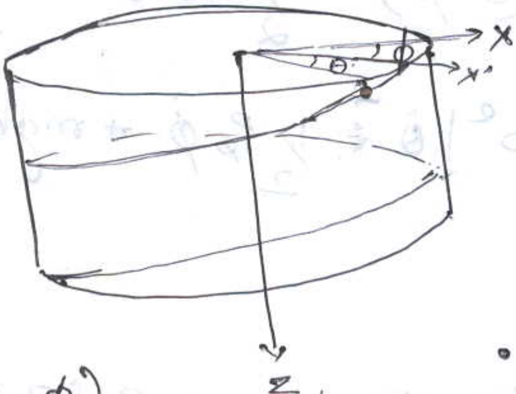
$$\ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = \frac{ml^2 \sin\theta \cos\theta \, \dot{\phi}^2 + mgl \sin\theta}{ml^2}$$

$$\ddot{\theta} = \sin\theta \cos\theta \, \dot{\phi}^2 + \frac{g}{l} \sin\theta$$

Eq of Motion. $\ddot{\theta} - \sin\theta \cos\theta \, \dot{\phi}^2 - g/l \sin\theta = 0$

Jamie (04d01022)
classical dynamics

Q. 48.



- φ → rotation of frame attached to the cyl.
- θ → rotation of particle w.r.t to frame attached to cyl.
- b → depends upon pitch of helix.

$x = r\cos(\theta-\phi)$

$y = r\sin(\theta-\phi)$

$z = b\theta$

$\Rightarrow \dot{x} = -r\sin(\theta-\phi)(\dot{\theta}-\dot{\phi})$

$\dot{y} = r\cos(\theta-\phi)(\dot{\theta}-\dot{\phi})$

$\dot{z} = b\dot{\theta}$

$\therefore L = \frac{1}{2}m[\dot{x}^2+\dot{y}^2+\dot{z}^2] - mgb\theta$

$L = \frac{1}{2}m[r^2(\dot{\theta}-\dot{\phi})^2 + b^2\dot{\theta}^2] - mgb\theta$

$\frac{\partial L}{\partial \dot{\theta}} = mr^2(\dot{\theta}-\dot{\phi}) + mb^2\dot{\theta} = P_\theta$ — ①

$\frac{\partial L}{\partial \dot{\phi}} = -mr^2(\dot{\theta}-\dot{\phi}) = P_\phi \Rightarrow \dot{\phi} = \frac{P_\phi}{mr^2} + \dot{\theta}$ — ②

$\dot{\theta} = \frac{P_\theta + P_\phi}{mb^2}$ — ③ [by substituting eq 2 in ①.

$\dot{\phi} = \frac{P_\phi}{mr^2} + \frac{P_\theta+P_\phi}{mb^2}$ — ④

Now.

$$H = P_\theta\dot{\theta} + P_\phi\dot{\phi} - \frac{1}{2}m\left[\xi^2(\dot{\theta}-\dot{\phi})^2 + b^2\dot{\theta}^2\right] + mgb\theta$$

$$= P_\theta\dot{\theta} + P_\phi\dot{\phi} + \frac{P_\phi}{2}(\dot{\theta}-\dot{\phi}) - \frac{1}{2}mb^2\dot{\theta}^2 + mgb\theta$$

$$\Rightarrow \left[P_\theta + \frac{P_\phi}{2} - \frac{1}{2}mb^2\right]\dot{\theta} + \frac{1}{2}P_\phi\dot{\phi} + mgb\theta \qquad -5$$

eq 3 & 4 in 5

$$\Rightarrow \left[P_\theta + \frac{P_\phi}{2} - \frac{1}{2}mb^2\right]\left[\frac{P_\theta + P_\phi}{mb^2}\right] + \frac{1}{2}P_\phi\left[\frac{P_\phi}{m\xi^2} + \frac{P_\theta + P_\phi}{mb^2}\right] + mgb\theta$$

$$= \frac{P_\theta^2 + \cancel{P_\phi \frac{P_\theta}{2}} - \frac{1}{2}mb^2 P_\theta + P_\phi P_\theta + \frac{P_\phi^2}{2} - \frac{1}{2}mb^2 P_\phi}{mb^2} + \frac{1}{2}\frac{P_\phi^2}{m\xi^2} + \frac{1}{2}\frac{\cancel{P_\phi P_\theta} + P_\phi^2}{mb^2} + mgb\theta$$

$+ mgb\theta$

$$= \frac{P_\theta^2 + P_\phi^2}{mb^2} - (P_\theta + P_\phi) + P_\theta$$

ignore

$$H \Rightarrow \frac{P_\theta^2 + P_\phi P_\theta}{mb^2} + \frac{P_\phi^2}{2m\xi^2} - \frac{(P_\theta + P_\theta)}{2} + mgb\theta$$

then

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{2P_\theta + P_\phi}{mb^2} - \frac{1}{2}$$

$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{m\xi^2} + \frac{P_\theta}{mb^2} - \frac{1}{2}$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = -mgb$$

$$\dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

---

continue.

$$\Rightarrow H = \frac{P_\theta^2 + \cancel{2} P_\theta P_\phi + \cancel{P_\phi^2}}{mb^2} - \left(\frac{P_\theta + P_\phi}{2}\right) - \frac{P_\phi^2}{2m\xi^2} + mgb\theta$$

$$= \frac{(P_\theta + P_\phi)P_\theta}{mb^2} - \frac{P_\phi^2}{2m\xi^2} - \left(\frac{P_\phi + P_\theta}{2}\right) + mgb\theta$$

$$\therefore \dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{2P_\theta + P_\phi}{mb^2} - \frac{1}{2}.$$

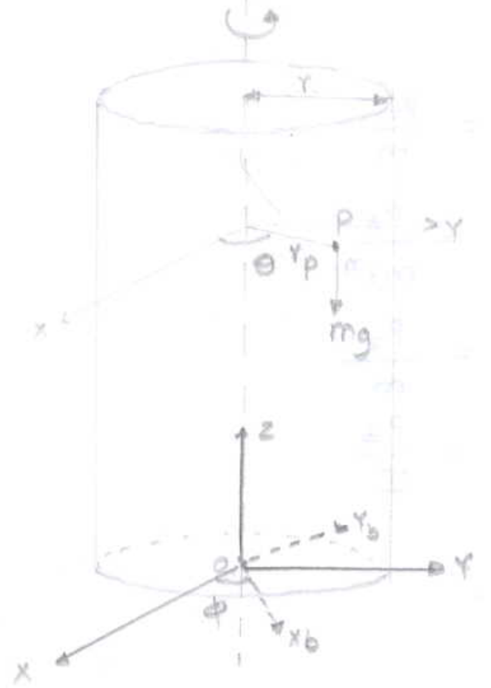
$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{mb^2} - \frac{P_\phi}{m\xi^2} - \frac{1}{2}.$$

$$\dot{P}_\theta = -mgb.$$

$$\dot{P}_\phi = 0.$$

//

Name: Barve Hrushikesh Arun
Roll No.: 07323003



Q. A particle of mass m moves under the action of gravity inside a smooth circular tube whose plane remains vertical. The tube is free to rotate about a vertical axis, passing through its centre, and has inertia I and radius r.

Write down Hamilton's equations for the system, using suitable independent generalised co-ordinates.

Ans.

1. The system of a particle and a circular tube, is shown in the above diagram.

Here,

r = radius of the tube

r_p = radius of the position vector of ~~par~~ the particle P.

θ = angle made by the particles position, with positive X axis.

ϕ = angle made by the co-ordinate frame attached to body i.e. the tube, with inertial reference frame.

z = z-co-ordinate of particle P in inertial reference frame.

2. So, in this system, there are 2 cases.

(i) particle inside the tube, not on its surface i.e. $r_p < r$

(ii) particle is on the tube's surface i.e. $r_p = r$

3. Case (i) : $r_p < r$

The generalised cordinates can be chosen as,

$q = (r, \theta, z, \phi)$

Then

$$L = T - V$$

$$= \frac{1}{2} m (\dot{r}^2 + r\dot{\theta}^2 + \dot{z}^2) + \frac{1}{2} I \dot{\phi}^2 - mgz$$

(Here, tube is assumed to be fixed, in such a way that, it just performs rotation)

4. Since, $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$,

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \therefore \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \therefore \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \therefore \dot{z} = \frac{p_z}{m}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi} \qquad \therefore \dot{\phi} = \frac{p_\phi}{I}$$

5. So, the Hamiltonian is given as,

$$H = \sum_{i=1}^{4} p_i \dot{q}_i - L$$

$$= \left(p_r \cdot \dot{r} + p_\theta \cdot \dot{\theta} + p_z \cdot \dot{z} + p_\phi \cdot \dot{\phi}\right) - L$$

$$= \left(\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_z^2}{m} + \frac{p_\phi^2}{I}\right) - \left(\frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{1}{2}\frac{p_z^2}{m} + \frac{1}{2}\frac{p_\phi^2}{I} - mgz\right)$$

$$= \frac{1}{2}\left(\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_z^2}{m} + \frac{p_\phi^2}{I}\right) + mgz$$

(So, H is nothing but total energy of the system, as the forces present in the system are conservative.)

6. Hamiltonian eq^{n}s are,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

So, Here,

$$\left.\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
\dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \\
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{I}
\end{aligned}\right\} \text{Eq}^n \text{ set (A)}$$

(These are the same equations, as obtained before in 4.)

and,

$$\left.\begin{aligned}
\dot{p}_r &= -\frac{\partial H}{\partial r} = -\frac{p_\theta^2}{2m}\frac{-2}{r^3} = \frac{p_\theta^2}{mr^3} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \\
\dot{p}_z &= -\frac{\partial H}{\partial z} = -mg \\
\dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = 0
\end{aligned}\right\} \text{Eq}^n \text{ set(B)}$$

The equations-sets (A) and (B) together form 8 first order eq^{n}s, which describe the system, which has 4 degrees of freedom.

7. <u>Case (ii)</u> : <u>$r_p = r$</u>

In this case, the generalised co-ordinates are $q = (\theta, z, \phi)$

So, $L = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{z}^2) + \frac{1}{2}I\dot{\phi}^2 - mgz$

and thus, Hamiltonian is,

$$H = \frac{1}{2}\left(\frac{p_\theta^2}{mr^2} + \frac{p_z^2}{m} + \frac{p_\phi^2}{I}\right) + mgz$$

and Hamiltonian eq^{n}s are,

$$\dot{\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{p}_\theta = 0$$

$$\dot{z} = \frac{p_z}{m}, \qquad \dot{p}_z = -mg$$

$$\dot{\phi} = \frac{p_\phi}{I}, \qquad \dot{p}_\phi = 0$$

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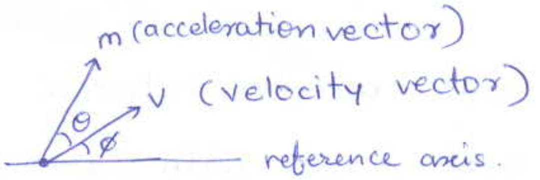
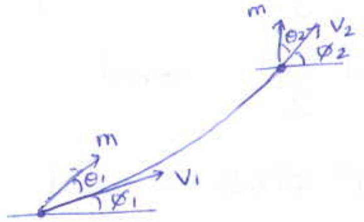
(37) Trajectory with shortest arc length to achieve $\Delta\phi = \phi_2 - \phi_1$



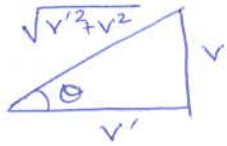


Functional to be minimised:

$$I = \int_0^{l} ds = \int_0^{t_f} v\, dt$$

kinematic equations: $\dfrac{dv}{dt} = m\cos\theta$, $\dfrac{d\phi}{dt} = \dfrac{m}{v}\sin\theta$.

$$\therefore\ dt = \frac{v\,d\phi}{m\sin\theta}$$



$$\frac{dv}{d\phi} = v' = \frac{m\cos\theta}{\frac{m}{v}\sin\theta} = v\cot\theta .$$

$$\therefore\quad \tan\theta = \frac{v}{v'} \quad , \quad \sin\theta = \frac{v}{\sqrt{v'^2+v^2}}$$

$$\therefore I = \int_0^{t_f} v\,dt = \int_{\phi_1}^{\phi_2} \frac{v^2 d\phi}{m\sin\theta} = \frac{1}{m}\int_{\phi_1}^{\phi_2} v\sqrt{v'^2+v^2}\, d\phi = \frac{1}{m}\int_{\phi_1}^{\phi_2} L(v,v')\,d\phi$$

We see that $L(v,v') = v\sqrt{v'^2+v^2}$ is independent of ϕ.

$\therefore$ Euler Lagrange equation reduces to Jacobi Integral being conserved. (as long as $v' \neq 0$)

$$\frac{d}{d\phi}\left(\frac{\partial L}{\partial v'}\cdot v' - L\right) = 0 \Rightarrow L - v'\frac{\partial L}{\partial v'} = \text{constant}.$$

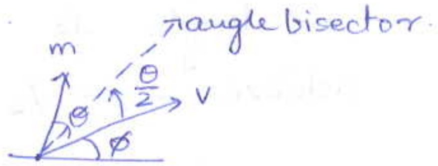
$$\therefore\quad v\sqrt{v'^2+v^2} - v'\left[\frac{v\cdot(2v')}{2\sqrt{v'^2+v^2}}\right] = \text{const.}$$

$$\therefore\quad \frac{v^3 + \cancel{v'^2 v} - \cancel{v'^2 v}}{\sqrt{v'^2+v^2}} = \text{const}$$

$$\therefore\ v^2 \cdot \frac{v}{\sqrt{v'^2+v^2}} = \text{const}.$$

$$\therefore\ \boxed{v^2\sin\theta = \text{const}}$$ along the length optimal trajectory

T.P.T : angle bisector between acc. & vel. vectors makes const angle with reference axis.



T.P.T : . . $\phi + \frac{\theta}{2} = \text{const.}$

We have $v^2 \sin\theta = \text{const.}$

$\therefore \quad 2v\dot{v}\sin\theta + v^2\cos\theta\,\dot{\theta} = 0$ (differenciating w.r.t. time)

$\therefore$ ~~2v~~ But $\dot{v} = m\cos\theta$, $\therefore \dot{v}\sin\theta = m\cos\theta\sin\theta = v\cos\theta\,\dot{\phi}$

$\therefore \quad 2v^2\cos\theta\,\dot{\phi} + v^2\cos\theta\,\dot{\theta} = 0$

$\therefore \quad 2v^2\cos\theta\left(\dot{\phi} + \frac{\dot{\theta}}{2}\right) = 0$ along trajectory.

$\rightarrow$ v ~~can~~ $\neq 0$, [If v is 0 all along trajectory $\Rightarrow$ stationary point. no change in heading]

~~$\therefore$ Either $\dot{\phi} + \frac{\dot{\theta}}{2} = 0$ $\Downarrow$ $\left(\phi + \frac{\theta}{2}\right) = \text{const.}$ Thus proved~~

$\rightarrow$ $\cos\theta \neq 0$ ~~or~~ If $\cos\theta = 0 \Rightarrow \sin\theta = \pm 1$, $\theta = \pm\frac{\pi}{2}$

$\therefore \dot{v} = 0$, $\dot{\phi} = \pm\frac{m}{v}$

$\therefore$ $\underline{v = \text{const}}$, $\dot{\phi} = \text{const}$, $\phi = at + b$

$\phi + \frac{\theta}{2} = at + b \pm \frac{\pi}{4}$

But problem is posed as, $\phi_i = \phi_1$, $\phi_f = \phi_2$,

$v(\phi_1) = v_i$, $v(\phi_2) = v_f$.

$\therefore v = \text{const}$, may not be a general solution unless $v_i = v_f$

may not satisfy original Euler Lagrange eq, since $v' = 0$

$\rightarrow \therefore \quad \dot{\phi} + \frac{\dot{\theta}}{2} = 0$ along trajectory.

i.e. $\phi + \frac{\theta}{2} = \text{const.}$

$\therefore$ angle bisector makes constant angle with reference axis

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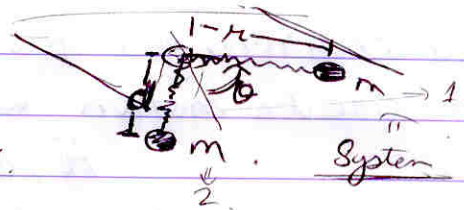
Mass m

unstretched length l_0 axial stiffness k.



System

Generalized co-ordinates : Polar co-ordinates of mass 1.

r, θ

Vertical position of mass 2

d.

Lagrangian of the system

$$L = T - V$$

$$= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 \quad \text{(KE mass 1)}$$

$$+ \frac{1}{2} m\dot{d}^2 - \frac{1}{2} k(r+d-l_0)^2 + mgd.$$

(KE mass 2) (Spring PE) (Gravitational PE)

$\theta \Rightarrow$ Ignorable co-ordinate $\frac{\partial L}{\partial \theta} = 0 \quad \frac{\partial L}{\partial \dot{\theta}} = P_\theta = m r^2 \dot{\theta}$

$$\dot{\theta} = P_\theta / m r^2 \tag{1}$$

Routhian reduction.

$$R = L - P_\theta \dot{\theta}$$

using (1)

$$= \left[\frac{1}{2} m(\dot{d}^2 + \dot{r}^2) - \frac{1}{2} k(r+d-l_0)^2 + mgd + \frac{1}{2} \frac{P_\theta^2}{m r^2}\right] - \frac{P_\theta^2}{m r^2}.$$

$$= \frac{1}{2} m(\dot{d}^2 + \dot{r}^2) + mgd - \frac{1}{2} k(r+d-l_0)^2 - \frac{P_\theta^2}{2 m r^2}.$$

System of equations for non ignorable co-ordinates.

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{d}}\right) - \frac{\partial R}{\partial d} = \frac{d}{dt}(m\dot{d}) - (mg - k(r+d-l_0)) = 0$$

$$\Longrightarrow \therefore m\ddot{d} - mg + k(r+d-l_0) = 0$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = \frac{d}{dt}(m\dot{r}) + (k(r+d-l_0) - \frac{P_\theta^2}{m r^3} = 0$$

$$\Longrightarrow \therefore m\ddot{r} + k(r+d-l_0) - \frac{P_\theta^2}{m r^3} = 0.$$

Initial conditions for first particle to perform circular motion.

for circular motion we require.

$$\ddot{r}, \ddot{d} = 0$$

$$\therefore \text{(i)} \quad mg = k(r+d-l_0)$$

$$\text{(ii)} \quad \frac{P_\theta^2}{mr^3} = k(r+d-l_0)$$

$$\therefore \quad P_\theta^2 = m^2 r^3 g$$

$$m^2 r^4 \dot{\theta}^2 = m^2 r^3 g$$

$$\dot{\theta}^2 = g/r \qquad \underline{\underline{\dot{\theta} = \sqrt{g/r}}}$$

Initial condition for circular motion of mass 1.

$$\dot{\theta} = \sqrt{g/r}.$$