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QUIZ 1, SOLUTIONS.

1). a) V_1 is a vector space, since sums & scalar multiples of symmetric matrices are symmetric.

V_2 is not a vector space since ~~$A \in V_2 \not\Rightarrow A \in V_2$~~.
$I \in V_2$ but $-I \notin V_2$.

b). V_2 has dimension $1+2+3 = 6$.

A basis for V_2 is given by the following collection of matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

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$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

c). f_1 is a linear operator since $\text{trace}(A_1 + A_2) = \text{tr}(A_1) + \text{tr}\, A_2$.
& $\text{trace}(\alpha A_1) = \alpha\, \text{trace}\, A_1$.

f_2 is not a linear operator since

~~det~~ $\det A_1 = 1$, $\det A_2 = -2$, but $\det(A_1 + A_2) = 0 \neq \det A_1 + \det A_2$
& for $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

d). $\mathcal{L} = f_1$. The range of $\mathcal{L}$ is $\mathbb{R}$.
The kernel of $\mathcal{L}$ is the set of symmetric matrices having zero trace. Since range $\mathcal{L} = \mathbb{R}$, rank $\mathcal{L} = 1$.

rank $\mathcal{L}$ + nullity $\mathcal{L}$ = $\dim V_1 = 6$.

Hence nullity $\mathcal{L} = 6 - 1 = 5$.

A basis for kernel $\mathcal{L}$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

2. Let $f_1(t) = e^t$, $f_2(t) = te^t$, $f_3(t) = t^2e^t$.

Suppose $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$.

putting $t=0$ yields $\alpha_1 = 0$.

$\therefore \alpha_2 te^t + \alpha_3 t^2 e^t = 0 \quad \forall t \in [0,1]$

$\Rightarrow \alpha_2 e^t + \alpha_3 te^t = 0 \quad \forall t \in (0,1]$. 2

Letting $t \to 0 \Rightarrow \alpha_2 = 0$.

$\therefore \alpha_3 = 0$.

Thus f_1, f_2, f_3 are linearly independent

3. Let $u^1 = 1$, $u^2 = s$, $u^3 = s^2$, $u^4 = s^3$.

$\mathcal{L}(u^1) = 2 = 2u^1$ 1½

$\mathcal{L}(u^2) = 3 + 2s = 3u^1 + 2u^2$

$\mathcal{L}(u^3) = 2 + 6s + 2s^2 = 2u^1 + 6u^2 + 2u^3$.

$\mathcal{L}(u^4) = \cancel{(12s) + 3(6s^2) + 2s^3} = 12u^2 + 18u^3 + 2u^4$

$6s + 9s^2 + 2s^3 = 2u^4 + 9u^3 + 6u^2$.

$\therefore$ The matrix representation of $\mathcal{L}$ in the basis $\{u^1, u^2, u^3, u^4\}$ is

$$\begin{bmatrix} 2 & 3 & 2 & \cancel{12}\,0 \\ 0 & 2 & 6 & \cancel{12}\,6 \\ 0 & 0 & 2 & \cancel{18}\,9 \\ 0 & 0 & 0 & \cancel{2}\,2 \end{bmatrix}.$$

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The matrix representation has rank 4. $\therefore$ $\mathcal{L}$ has rank 4. 1.

(OR Note that the eqn. $\mathcal{L}(p)=0$ has only exponential solutions,

$\therefore$ kernel $\mathcal{L}$ has no elements in $\mathbb{R}_3[s]$.

4. The set of invertible matrices do not form a field because the sum of invertible matrices need not be invertible

e.g. $I + -I$. 2.

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Quiz 2, Solutions

1. $F(t) = \begin{bmatrix} e^{t} & e^{-t} \\ e^{-t} & e^{t} \end{bmatrix}$, $F(s) = \begin{bmatrix} e^{s} & e^{-s} \\ e^{-s} & e^{s} \end{bmatrix}$.

$$F(t).F(s) = \begin{bmatrix} e^{t+s} + e^{-(t+s)} & e^{t-s} + e^{s-t} \\ e^{s-t} + e^{t-s} & e^{-(t+s)} + e^{(t+s)} \end{bmatrix}.$$

$$F(t+s) = \begin{bmatrix} e^{t+s} & e^{-(t+s)} \\ e^{-(t+s)} & e^{(t+s)} \end{bmatrix} \neq F(t).F(s).$$

$\therefore$ F is not a matrix exponential.

OR $$F^{-1}(t) = \frac{1}{e^{2t} - e^{-2t}} \begin{bmatrix} e^{t} & -e^{-t} \\ -e^{-t} & e^{t} \end{bmatrix}.$$

$$F(-t) = \begin{bmatrix} e^{-t} & e^{t} \\ e^{t} & e^{-t} \end{bmatrix} \neq F^{-1}(t).$$

2. Let $F(t) = e^{3t} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. $\dot{F}(t) = 3F(t)$.

$$AF(t) = e^{3t} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = e^{3t} \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \dot{F}(t).$$

But $$F(0) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \neq I.$$

$\therefore F(t) \neq e^{At}$

3 $\dot{x}_1 = 2x_2, \; \dot{x}_2 = 2x_1 \Rightarrow \quad \dot{x} = Ax$, where

$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$. $\cancel{A^2 = 4I \quad A^3 = 8A}$. $= 2P$.

~~in general,~~ $\cancel{A^{2k} = 2^{2k} I, \; A^{2k+1} = 2^{2k+1} A}$.

$P^2 = I, \; P^3 = P, \; \ldots \; \therefore P^{2k} = I, \quad P^{2k+1} = P$.

$\therefore A^{2k} = 2^{2k} I, \; A^{2k+1} = 2^{2k+1} P$. |

$\therefore e^{At} = I\left[1 + \frac{1}{2!} 2^2 t^2 + \frac{1}{4!} 2^4 t^4 \ldots\right]$

$+ P\left[2t + \frac{2^3 t^3}{3!} + \ldots\right]$.

$= I\left(\frac{e^{2t} + e^{-2t}}{2}\right) + P\left(\frac{e^{2t} - e^{-2t}}{2}\right)$.

$= \begin{bmatrix} \frac{e^{2t} + e^{-2t}}{2} & \frac{e^{2t} - e^{-2t}}{2} \\ \frac{e^{2t} - e^{-2t}}{2} & \frac{e^{2t} + e^{-2t}}{2} \end{bmatrix}$. |

If $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the $x(t) = e^{At} x(0) = \frac{1}{2}\begin{bmatrix} e^{2t} + e^{-2t} \\ e^{2t} - e^{-2t} \end{bmatrix}$. |

4. $A - 2I = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$. rank $(A-2I) = 2 < 4$. |

$\therefore 2$ is an eigenvalue.

Since nullity $(A-2I) = 2$, geometric multiplicity $= 2$. |

$(A-2I)^2 = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$. rank $(A-2I)^2 = 1$.

~~S that it A is~~

4 $(A-2I)^3 = 0$.

$\therefore$ we conclude that algebraic multiplicity of $\lambda = 2$ is 4.

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The Jordan block associated with $\lambda=2$ is

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

5. $A^TA = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$. $\det A^TA = 8-4 = 4 > 0$.

$\therefore$ By Sylvester's criterion, $A^TA > 0$.

$A + A^T = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$. $A+A^T-\lambda I = \begin{bmatrix} -\lambda & -1 \\ -1 & -(\lambda+2) \end{bmatrix}$

$\det(A+A^T-\lambda I) = +\lambda(\lambda+2) - 1 = 0$ i.e $\lambda^2 + 2\lambda - 1 = 0$.

$\lambda = \dfrac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$. $\Rightarrow$ $A+A^T$ is indefinite

$A^TP_1 + P_1A = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$. — -ve semidefinite.

$A^TP_2 + P_2A = -\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ $\det. \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} > 0$.

$\therefore$ $A^TP_2 + P_2A$ is -ve definite.

3 (Alternative solution).

$$(s\underline{I}-A) = \begin{bmatrix} s & -2 \\ -2 & s \end{bmatrix} \quad (s\underline{I}-A)^{-1} = \begin{bmatrix} s & 2 \\ 2 & s \end{bmatrix} \frac{1}{s^2-4}$$

$$\frac{2}{s^2-4} = \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+2}\right] \qquad \frac{s}{s^2-4} = \frac{1}{2}\left[\frac{1}{s+2} + \frac{1}{s-2}\right].$$

$\therefore$ ~~$e^{At} = \mathcal{L}(s\underline{I}-A)^{-1} =$~~

$$e^{At} = \mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} \frac{e^{-2t}+e^{2t}}{2} & \frac{e^{2t}-e^{-2t}}{2} \\ \frac{e^{2t}-e^{-2t}}{2} & \frac{e^{-2t}+e^{2t}}{2} \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \quad x(t) = e^{At}x(0) = \frac{1}{2}\begin{bmatrix} e^{2t}+e^{-2t} \\ e^{2t}-e^{-2t} \end{bmatrix}.$$

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Quiz 3, Solutions

1. $P(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau}\, d\tau.$ Clearly, $P(0) = 0$.

$\dot P(t) = e^{A\tau} BB^T e^{A^T\tau}$.

$$AP + PA^T + BB^T = \int_0^t \left[A e^{A\tau} BB^T e^{A^T\tau} + e^{A\tau} BB^T e^{A^T\tau} A^T \right] d\tau + BB^T$$

$$= \int_0^t \frac{d}{d\tau} \left(e^{A\tau} BB^T e^{A^T\tau} \right) d\tau + BB^T$$

$$= e^{At} BB^T e^{A^T t} - BB^T + BB^T = \dot P(t)$$

Thus $\dot P = AP + PA + BB^T$ & $P(0) = 0$.

2. Suppose $A^T P + PA + 2\mu P = -Q$, $P > 0$, $Q > 0$, $\mu > 0$.
Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with eigenvector $v \in \mathbb{C}^n$. Then $Av = \lambda v$ & $v^* A^T = \bar\lambda v^*$.

$$\therefore v^*(A^T P + PA + 2\mu P) v = \bar\lambda v^* P v + \lambda v^* P v + 2\mu v^* P v$$
$$= -v^* Q v^*$$

$$\therefore (\lambda + \bar\lambda) v^* P v = -2\mu v^* P v - v^* Q v.$$

Since $v^* P v > 0$ & $v^* Q v > 0$, we conclude that

$$2 \operatorname{Re} \lambda = \lambda + \bar\lambda < -2\mu.$$

$$\therefore \operatorname{Re} \lambda < -\mu.$$

Since λ was chosen arbitrarily, ~~we~~ the conclusion follows.

3. The controllability matrix is

$$C = \begin{bmatrix} 1 & 2 & +4 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$ which has rank 2.

The system is not controllable.

$$\text{range } C = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Since $x_f = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \notin \text{range } C$, it follows that there exist no input that steers $x_i = 0$ to x_f.

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 2 & -1 \\ 9 & 4 & 13 \end{bmatrix}$$

If x_1 & x_2 have to yield the same outputs, then $Ce^{At}(x_1 - x_2) = 0$, ie $x_1 - x_2 \in$ ~~AE~~ kernel $\mathcal{O}$.

$$x_1 - x_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \& \quad \mathcal{O}(x_1 - x_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\therefore$ Outputs generated by x_1 & x_2 are the same.

4. $(A - 2I) = \begin{bmatrix} 0 & 1 & 1 \\ 5 & 1 & 6 \\ -5 & -1 & -6 \end{bmatrix}$. which has rank 2.

$\therefore$ 2 is an eigenvalue of A.

The only left eigenvector is $v = \cancel{\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}}\ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}^T$

$vB = 0$ $\therefore$ eigenvalue 2 is not controllable.

$(A + 3I) = \begin{bmatrix} 5 & 1 & 1 \\ 5 & 6 & 6 \\ -5 & -1 & -1 \end{bmatrix}$. which has rank 2.

$\therefore -3$ is an eigenvalue.

The right eigenvector of -3 is $v = \cancel{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}.\ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

$Cv = -1 \neq 0$.

$\therefore -3$ is observable.