Controllability of Nonlinear Systems

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1 Introduction

The purpose of these notes is to outline the main ideas and results in theory of controllability of nonlinear systems. To simplfy the exposition, we mainly focus on driftless systems. Instead of proofs, we adopt intuitive reasoning and examples to show how the notions, ideas and results encountered in this area arise naturally.

These notes assume a familiarity with the notions of differentiable manifolds, submanifolds, tangent spaces, vector fields, flows, and Lie brackets.

2 Input-affine systems

To describe the main ideas underlying nonlinear controllability theory, we will consider a control system whose state evolves on an n-dimensional smooth connected manifold $\mathcal{M}$. We will assume that the evolution of the state of the system is described by an *input-affine* differential equation of the form

$$\dot{x}(t) = F_0\big(x(t)\big) + F_1\big(x(t)\big)u_1(t) + \cdots + F_m\big(x(t)\big)u_m(t), \tag{1}$$

where $x(t) \in \mathcal{M}$ is the state of the system at time $t \in \mathbb{R}$, $u_1, \ldots, u_m$ are real-valued input functions, and $F_0, \ldots, F_m$ are vector fields on $\mathcal{M}$. The vector fields $F_1, \ldots, F_m$ associated with inputs are the *control vector fields*, while the vector field F_0 is the drift vector field.

We will assume that each input function is piecewise constant, and can take any value in an open interval $I \subseteq \mathbb{R}$ containing zero. Let $\mathcal{D} = \{F_0 + F_1u_1 + \cdots + F_mu_m : (u_1, \ldots, u_m) \in \mathbb{R}^m\}$ be the set of all vector fields that one can obtain on the righthandside of (1) by using all possible constant input combinations. Then, in response to a set of constant inputs, the state of the system evolves along an integral curve of one of the vector fields in $\mathcal{D}$. More generally, the state trajectory generated by a piecewise constant input vector will be composed of several segments, each of which lies along the integral curve of one of the vector fields in $\mathcal{D}$. Given $z \in \mathcal{M}$, the *reachable set* $\mathcal{R}(z)$ from z is the set of all states that can be reached by starting at z and using all possible piecewise constant inputs. The reachable set from z is thus given by

$$\mathcal{R}(z) = \left\{ \phi_{t_k}^{X_k} \circ \cdots \circ \phi_{t_1}^{X_1}(z) : k \geq 0,\ X_1, \ldots, X_k \in \mathcal{D},\ t_1, \ldots, t_k \geq 0 \right\}. \tag{2}$$

The system (1) is *controllable* if its state can be steered from any initial point to any desired final point in $\mathcal{M}$, that is, if $\mathcal{R}(z) = \mathcal{M}$, for every $z \in \mathcal{M}$.

Our first example below illustrates a typical way in which a system may fail to be controllable.

Example 2.1. Let $\mathcal{M} = \mathbb{R}^3$, and suppose

$$F_0(x) = \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}, \quad F_1(x) = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}.$$

It is easy to check that $x^{\mathrm{T}} F_0(x) = x^{\mathrm{T}} F_1(x) = 0$, for all $x \in \mathbb{R}^3$. Thus, $\dfrac{\mathrm{d}}{\mathrm{d}t} x(t)^{\mathrm{T}} x(t) = 2x(t)^{\mathrm{T}} \dot{x}(t) = 0$, so that $x(t)^{\mathrm{T}} x(t) = x(0)^{\mathrm{T}} x(0)$, no matter what the input. Thus, for this example $\mathcal{R}(z) \subseteq \left\{ w \in \mathbb{R}^3 : w^{\mathrm{T}} w = z^{\mathrm{T}} z \right\}$. □

Not only is the system considered in Example 2.1 not controllable, its reachable sets are actually constrained to lie in a lower dimensional submanifold of $\mathcal{M}$. Clearly, a necessary condition for controllability is that the reachable sets should not be restricted to lower dimensional submanifolds of $\mathcal{M}$. One way of ensuring this is to require that each reachable set should contain an open subset of $\mathcal{M}$. This idea motivates our next definition. The system (1) is *accessible* if $\mathcal{R}(z)$ has a nonempty interior in $\mathcal{M}$ for every $z \in \mathcal{M}$.

The basic issues in controllability theory are

1. To characterize the reachable sets, and

2. To find conditions on $F_0, \ldots, F_m$ under which the system is controllable or at least accessible.

We will treat these issues first for drift free systems in the next section.

3 Drift-free systems

The system (1) is *drift free* if the drift vector field F_0 is zero. In this case (1) becomes

$$\dot{x}(t) = F_1\big(x(t)\big)u_1(t) + \cdots + F_m\big(x(t)\big)u_m(t), \tag{3}$$

while $\mathcal{D} = \{F_1u_1 + \cdots + F_mu_m : (u_1, \ldots, u_m) \in \mathbb{R}^m\} = \mathrm{span}\{F_1, \cdots, F_m\}$. Thus the set of vector fields $\mathcal{D}$ is a vector space over $\mathbb{R}$. At each $x \in \mathcal{M}$, the vector fields in $\mathcal{D}$ span the linear subspace

$$\Delta_{\mathcal{D}}(x) \triangleq \mathrm{span}\{X(x) : X \in \mathcal{D}\} = \mathrm{span}\{F_1(x), \cdots, F_m(x)\}$$

of the tangent space $T_x\mathcal{M}$ of $\mathcal{M}$ at x. The collection $\Delta_{\mathcal{D}} \triangleq \cup_{x\in\mathcal{M}}\Delta_{\mathcal{D}}(x)$ of all these subspaces is the *distribution* spanned by $\mathcal{D}$.

3.1 Distributions and Integrability

In general, a distribution on $\mathcal{M}$ is a collection of subspaces of tangent spaces of $\mathcal{M}$ that, loosely speaking, depend smoothly on the base point x. For example, the collection of subspaces $\Delta(x) \overset{\text{def}}{=} \{v \in \mathbb{R}^3 : v^{\mathrm{T}}x = 0\}$ form a distribution on $\mathcal{M} \overset{\text{def}}{=} \mathbb{R}^3\backslash\{0\}$. For every $x \in \mathbb{R}^3\backslash\{0\}$, $\Delta(x)$ is simply the two-dimensional tangent space at x to the sphere passing through x and centered at zero. The two-dimensional distribution has a special property: for every $x \in \mathcal{M}$, there exists a smooth two-dimensional submanifold $\mathcal{N}_x$ (namely, the sphere passing through x and centered at zero) of $\mathcal{M}$ such that the tangent space to $\mathcal{N}_x$ at each $z \in \mathcal{N}_x$ is $\Delta(z)$. This property is called integrability.

An integral manifold of Δ is a submanifold $\mathcal{N}$ of $\mathcal{M}$ such that $T_x\mathcal{N} = \Delta(x)$ for every $x \in \mathcal{N}$. A distribution Δ is *integrable* if, for every $x \in \mathcal{M}$, there exists an integral manifold of Δ passing through x. If Δ is integrable, there exists a maximal integral manifold of Δ through every point $x \in \mathcal{M}$.

The controllability of the drift-free system (3) depends on the integrability of the distribution $\Delta_{\mathcal{D}}$. Indeed, if $x : \mathbb{R} \to \mathcal{M}$ is a state trajectory of (3), then $\dot{x}(t) \in \Delta_{\mathcal{D}}(x(t))$ for every $t \in \mathbb{R}$. This means that, if $\Delta_{\mathcal{D}}$ is integrable, then $\dot{x}(t)$ is tangent to the maximal integral manifold of $\Delta_{\mathcal{D}}$ at $x(t)$, and hence the trajectory is contained in the maximal integral manifold of $\Delta_{\mathcal{D}}$. In cases where the integral manifolds of $\Delta_{\mathcal{D}}$ have dimension less than n, the integrability of $\Delta_{\mathcal{D}}$ rules out controllability.

The integrability of a distribution depends on its involutivity. A distribution on $\mathcal{M}$ is *involutive* if, for every pair of vector fields X, Y satisfying $X(x), Y(x) \in \Delta(x)$ for every $x \in \mathcal{M}$, the vector field $[X, Y]$ satisfies the same condition. The relationship between integrability and involutivity is given by Frobenius' theorem [1, Thm. 8.3].

Theorem 3.1 (Frobenius). *Suppose a distribution Δ has constant dimension. Then Δ is integrable if and only if Δ is involutive.*

Reference [2] contains other sufficient as well as necessary and sufficient conditions for integrability, some of which apply even when the distribution does not have constant rank.

Let us apply Frobenius' theorem to a few driftless systems.

Example 3.1. Let $\mathcal{M} = \mathbb{R}^3 \backslash \{0\}$ and suppose

$$F_1(x) = \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}, \quad F_3(x) = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}.$$

The three vector fields F_1, F_2 and F_3 span a two-dimensional subspace of $\mathbb{R}^3$ at every $x \in \mathcal{M}$ (can you show this?). Hence the distribution $\Delta_{\mathcal{D}}$ is two dimensional. An easy calculation yields $[F_1, F_2] = -F_3$, $[F_2, F_3] = -F_1$ and $[F_3, F_1] = -F_2$. Hence the distribution $\Delta_{\mathcal{D}}$ spanned by the vector fields F_1, F_2 and F_3 is involutive. By Frobenius' theorem, the distribution $\Delta_{\mathcal{D}}$ is integrable. Indeed, for every $x \in \mathcal{M}$, all tangent vectors in $\Delta_{\mathcal{D}}(x)$ are orthogonal to x. The distribution $\Delta_{\mathcal{D}}$ is simply the collection Δ of all planes tangent to spheres centered at the origin which we saw a little earlier. The maximal integral manifold of $\Delta_{\mathcal{D}}$ passing through a given $x \in \mathcal{M}$ is simply the sphere passing through x and centered at zero. This is consistent with our conclusion in Example 2.1 that the reachable set from any $x \in \mathcal{M}$ is contained in the sphere passing through x and centered at zero. The system is not accessible. Note, however, that we do not know what the reachable set is. It could, for instance, be the whole sphere, or some subset of it. In the next subsection, we will see results that will help us ascertain what the reachable set of this system actually is. □

Example 3.2. Let $\mathcal{M} = \mathbb{R}^4 \backslash \{0\}$ and let

$$F_1(x) = \begin{bmatrix} x_4 \\ x_3 \\ -x_2 \\ -x_1 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} x_3 \\ -x_4 \\ -x_1 \\ x_2 \end{bmatrix}.$$

Note that the vector fields F_1 and F_2 are nonzero and mutually orthogonal at every point $x \in \mathcal{M}$. Consequently, the distribution $\Delta_{\mathcal{D}}$ is two dimensional at every point in $\mathcal{M}$. An easy computation yields

$$[F_1, F_2](x) = 2 \begin{bmatrix} x_2 \\ -x_1 \\ x_4 \\ -x_3 \end{bmatrix}.$$

The vector $[F_1, F_2](x)$ is nonzero and orthogonal to each of the vectors $F_1(x)$ and $F_2(x)$ for every $x \in \mathcal{M}$. Hence the distribution $\Delta_{\mathcal{D}}$ is not involutive. Frobenius' theorem implies that $\Delta_{\mathcal{D}}$ is not integrable. We can immediately conclude that the reachable set from any point in $\mathcal{M}$ is not restricted to lie in any two-dimensional submanifold of $\mathcal{M}$. However, the system is not accessible. Indeed, for every $x \in \mathcal{M}$, $x^{\mathrm{T}} F_1(x) = x^{\mathrm{T}} F_2(x) = 0$. Consequently, the vectors $F_1(x)$ and $F_2(x)$ lie in the tangent space to the sphere $\{w \in \mathbb{R}^4 : w^{\mathrm{T}} w = x^{\mathrm{T}} x\}$ passing through x and centered at zero. We can thus conclude that the reachable set from $x \in \mathcal{M}$ is contained in a three-dimensional submanifold, namely the sphere passing through x and centered at zero. $\square$

Example 3.3. The figure below shows a differential drive axle that can move on a plane. Let x_1 and x_2 denote the Cartesian coordinates of the center of the axle, and let x_3 denote the angular orientation of the axle. The combined rotational motions of the two wheels impart a linear velocity u_1 transverse to the axle, and an angular velocity u_2 to the axle as shown in the figure. We consider a kinematic model in which the wheel motions can be controlled to achieve any desired values for u_1 and u_2. The equations

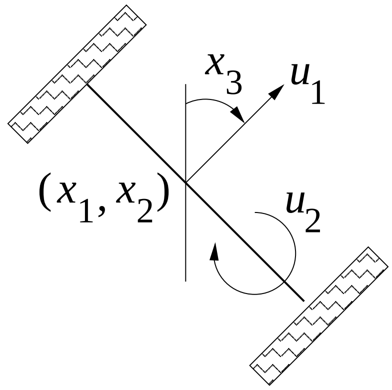

Figure 1: A kinematic axle on a plane

describing such a kinematic model are given by (3) with $\mathcal{M} = \mathbb{R}^3$ and

$$F_1(x) = \begin{bmatrix} \sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The vector field F_1 represents forward motion transverse to the axle, while F_2 represents steering.

The vector fields F_1 and F_2 are linearly independent everywhere and hence the distribution $\Delta_{\mathcal{D}}$ is two-dimensional. Is it integrable? Our experience suggests that it is not, because integrability of $\Delta_{\mathcal{D}}$ would imply that certain states, that is, combinations of positions and orientations are inaccessible to the axle, whereas we can easily visualize combinations of forward and rotational motions that steer the axle to any desired state. Indeed,

$$[F_1, F_2](x) = \begin{bmatrix} -\cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix}.$$

The vector $[F_1, F_2](x)$ is orthogonal to $F_1(x)$ and $F_2(x)$ for every $x \in \mathcal{M}$. Hence the distribution $\Delta_{\mathcal{D}}$ is not involutive. Frobenius' theorem implies that $\Delta_{\mathcal{D}}$ is not integrable, that is, reachable sets are not restricted to two-dimensional submanifolds. Is the system controllable? The next subsection will tell. □

3.2 Orbits and Controllability

The *orbit* of a point $x \in \mathcal{M}$ under the set of vector fields $\mathcal{D}$ is the set

$$\left\{\phi_{t_k}^{X_k} \circ \cdots \circ \phi_{t_1}^{X_1}(x) : k \geq 0,\ X_1, \ldots, X_k \in \mathcal{D},\ t_1, \ldots, t_k \in \mathbb{R}\right\},$$

that is, the orbit of x is the set of all points that can be reached by starting from x and travelling along the integral curves of vector fields *either forward or backward in time* in all possible sequences. Note that, in general the orbit of x is different from the reachable set from x, because points in the reachable set are those points which can be reached by travelling along the integral curves of vector fields in $\mathcal{D}$ going *only forward* in time. Clearly the reachable set from x is contained in the orbit of x, although the converse is not true in general.

Consider the drift-less system (3) once again. Because of the absence of a drift term, the sign of $\dot{x}$ can be reversed by reversing the signs of the inputs. Said another way, if $X \in \mathcal{D}$, then $-X \in \mathcal{D}$ as well. Also, going backward in time along the flow of X is the same as going forward in time along the flow of $-X$, that is, $\phi_{-t}^{X} = \phi_t^{-X}$ for every $t > 0$. These two facts combine to imply that

$$\left\{\phi_{t_k}^{X_k} \circ \cdots \circ \phi_{t_1}^{X_1}(x) : k \geq 0,\ X_1, \ldots, X_k \in \mathcal{D},\ t_1, \ldots, t_k \geq 0\right\}$$
$$= \left\{\phi_{t_k}^{X_k} \circ \cdots \circ \phi_{t_1}^{X_1}(z) : k \geq 0,\ X_1, \ldots, X_k \in \mathcal{D},\ t_1, \ldots, t_k \in \mathbb{R}\right\}.$$

In other words, the reachable set from X is the orbit of x.

To learn more about orbits, let $\mathcal{A}$ be a set of vector fields on $\mathcal{M}$, and consider $X, Y \in \mathcal{D}$, $x \in \mathcal{M}$ and $t \in \mathbb{R}$. Then $\phi_t^{X+Y}(x)$ is the limit of a sequence whose nth term is [3, Cor. 2.1.27]

$$x_n = \left(\phi_{t/n}^{X} \circ \phi_{t/n}^{Y}\right)^n (x).$$

Each element x_n is clearly in the orbit of x under $\mathcal{A}$. Hence $\phi_t^{X+Y}(x)$ is in the closure of the orbit of x. Next, $\phi_t^{[X,Y]}(x)$ is the limit of a sequence whose nth term is [3]

$$z_n = \left(\phi_{\sqrt{t/n}}^{-Y} \circ \phi_{\sqrt{t/n}}^{-X} \circ \phi_{\sqrt{t/n}}^{Y} \circ \phi_{\sqrt{t/n}}^{X}\right)^n (x). \tag{4}$$

Once again each element z_n of the sequence is in the orbit of x under $\mathcal{A}$. Hence $\phi_t^{[X,Y]}(x)$ is in the closure of the orbit of x.

The discussion above suggests that the orbit of a point under a set of vector fields $\mathcal{A}$ remains essentially unchanged if we expand $\mathcal{A}$ by recursively including in $\mathcal{A}$ the sums and brackets of vector fields in $\mathcal{A}$. The set of vector fields thus generated is $\mathcal{L}(\mathcal{A})$, the Lie algebra generated by $\mathcal{A}$.

In general, a Lie algebra of vector fields is a set of vector fields that is a vector space over $\mathbb{R}$ and is closed under Lie brackets. The Lie algebra generated by $\mathcal{A}$ is simply the smallest Lie algebra of vector fields that contains $\mathcal{A}$. The elements of $\mathcal{L}(\mathcal{A})$ are linear combinations of Lie brackets involving vector fields in $\mathcal{A}$.

Let us return to the driftless system (3) once again with its set of associated vector fields $\mathcal{D}$. Consider the distribution $\Delta_{\mathcal{L}(\mathcal{D})}$ spanned by $\mathcal{L}(\mathcal{D})$ and let $x \in \mathcal{M}$. Clearly $\Delta_{\mathcal{L}(\mathcal{D})}(x)$ contains all tangent vectors in $\Delta_{\mathcal{D}}(x)$, since $\mathcal{D} \subseteq \mathcal{L}(\mathcal{D})$. Since $\mathcal{L}(\mathcal{D})$ is a Lie algebra, $\Delta_{\mathcal{L}(\mathcal{D})}$ is involutive, even if $\Delta_{\mathcal{D}}$ is not. Hence Frobenius' theorem implies that $\Delta_{\mathcal{L}(\mathcal{D})}$ is integrable if its dimension at all points is the same. The orbit of a point under vector fields in $\mathcal{L}(\mathcal{D})$ is contained in the maximal integral manifold of $\mathcal{L}(\mathcal{D})$ passing through that point. On the other hand, the orbit of a point under $\mathcal{D}$ is clearly contained in the orbit of that point under the larger set of vector fields $\mathcal{L}(\mathcal{D})$. Hence we have

$$\begin{aligned}\mathcal{R}(x) = \text{orbit of } x \text{ under } \mathcal{D} &\subseteq \text{orbit of } x \text{ under } \mathcal{L}(\mathcal{D}) \\ &\subseteq \text{ maximal integral manifold of } \mathcal{L}(\mathcal{D}) \text{ through } x.\end{aligned}$$

Chow's theorem below essentially states that the inclusions above can in fact be replaced by equalities if $\Delta_{\mathcal{L}(\mathcal{D})}$ has the same dimension everywhere.

Theorem 3.2 (Chow). *Suppose $\Delta_{\mathcal{L}(\mathcal{D})}$ has the same dimension at all points. Then the orbit of x under $\mathcal{D}$ is the maximal integral manifold of $\Delta_{\mathcal{L}(\mathcal{D})}$.*

Chow's theorem allows us to determine whether a drift-free system is controllable or not in the following way. Start with the collection of vector fields $F_1, \ldots, F_m$. Compute succeedingly higher order Lie brackets of this vector fields and include them in the collection. At each step, check the dimension of the distribution spanned by the collection of vector fields. This dimension will be nondecreasing. If this dimension eventually reaches n at all points, then the integral manifold of $\Delta_{\mathcal{L}(\mathcal{D})}$ is $\mathcal{M}$ itself, and by Chow's theorem, the system is controllable. On the other hand, if the dimension reaches a value less than n at all points, then $\Delta_{\mathcal{L}(\mathcal{D})}$ has a lower dimensional integral manifold through each point, and Chow's theorem implies that the system is not controllable. Thus we have the following corollary.

Corollary 3.1. *Suppose* $\dim\Delta_{\mathcal{L}(\mathcal{D})}(x) = k$ *for every* $x \in \mathcal{M}$. *Then the system (3) is controllable if and only if* $k = n$.

Chow's theorem assumes that $\Delta_{\mathcal{L}(\mathcal{D})}$ has the same dimension at all points. This assumption is not required if the control vector fields are real analytic [5]. If the assumptions of constant dimensionality and analyticity both fail to hold, then the orbits are integral manifolds of a bigger distribution constructed by including additional directions in $\Delta_{\mathcal{L}(\mathcal{D})}$. See [2] for details. Finally, the condition that $\Delta_{\mathcal{L}(\mathcal{D})}$ have the dimension n at all points is often called the rank condition, or the Lie algebraic rank condition (LARC).

Example 3.4. Recall that, in Example 3.1, the distribution $\Delta_{\mathcal{D}}$ had the constant dimension two everywhere on $\mathcal{M} = \mathbb{R}^3\backslash\{0\}$. The distribution $\Delta_{\mathcal{D}}$ was involutive, so that $\Delta_{\mathcal{D}} = \Delta_{\mathcal{L}(\mathcal{D})}$. Moreover, the integral manifolds of $\Delta_{\mathcal{D}}$ were spheres centered at zero. Hence, Chow's theorem implies that the orbits of $\Delta_{\mathcal{D}}$ are spheres centered at zero. In other words, starting from any point $x \in \mathcal{M}$, it is possible to reach any other point on the sphere passing through x using suitable control inputs. Thus $\mathcal{R}(x) = \{z \in \mathcal{M} : z^{\mathrm{T}}z = x^{\mathrm{T}}x\}$ for all $x \in \mathcal{M}$. $\square$

Example 3.5. Recall that, in Example 3.2, for every $x \in \mathcal{M} = \mathbb{R}^3\backslash\{0\}$,

$$F_1(x) = \begin{bmatrix} x_4 \\ x_3 \\ -x_2 \\ -x_1 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} x_3 \\ -x_4 \\ -x_1 \\ x_2 \end{bmatrix}, \quad [F_1, F_2](x) = 2\begin{bmatrix} x_2 \\ -x_1 \\ x_4 \\ -x_3 \end{bmatrix}.$$

It is easy to check that, for every $x \in \mathcal{M}$,

1. $[F_1, [F_1, F_2]](x) = 4F_2(x)$, $[F_2, [F_1, F_2]](x) = -4F_1(x)$.
2. The three vectors $F_1(x)$, $F_2(x)$ and $[F_1, F_2](x)$ are orthogonal to x.

3. The three vectors $F_1(x)$, $F_2(x)$ and $[F_1, F_2](x)$ are linearly independent.

The statement 1) above implies that the distribution $\Delta_{\mathcal{L}(\mathcal{D})}$ is the distribution spanned by the vector fields F_1, F_2 and $[F_1, F_2]$. Statement 2) implies that the integral manifolds of $\Delta_{\mathcal{L}(\mathcal{D})}$ are spheres centered at zero. Statement 3) implies that $\Delta_{\mathcal{L}(\mathcal{D})}$ has the constant dimension three everywhere on $\mathcal{M}$. Hence Chow's theorem says that the orbits of $\Delta_{\mathcal{D}}$ are three-dimensional spheres centered at zero. In other words, $\mathcal{R}(x) = \{z \in \mathcal{M} : z^{\mathrm{T}} z = x^{\mathrm{T}} x\}$ for all $x \in \mathcal{M}$. □

Example 3.6. Recall that, in Example 3.3, we had

$$F_1(x) = \begin{bmatrix} \sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [F_1, F_2](x) = \begin{bmatrix} -\cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix}.$$

It is easy to check that the vector fields F_1, F_2 and $[F_1, F_2]$ are linearly independent everywhere on $\mathcal{M} = \mathbb{R}^3$. Hence the distribution $\Delta_{\mathcal{L}(\mathcal{D})}$ is involutive and has constant dimension three everywhere on $\mathcal{M}$. The integral manifold of $\Delta_{\mathcal{L}(\mathcal{D})}$ is $\mathcal{M}$ itself, since $\Delta_{\mathcal{L}(\mathcal{D})}(x) = T_x\mathcal{M}$ at every $x \in \mathcal{M}$. Chow's theorem implies that the orbit of $\Delta_{\mathcal{D}}$ (equivalently, the reachable set of (3)) is all of $\mathcal{M}$.

In plain English, Chow's theorem guarantees that the axle can be steered to reach any point in the plane with any desired orientation. Notice that the vector field $[F_1, F_2]$ represents sideways motion along the axle axis, which is not ordinarily possible for the axle. However, equation (4) suggests that it is possible to approximately "simulate" such sideways motion by switching between forward and steering motion. The ability of the system to simulate forbidden motions is what makes the distribution $\Delta_{\mathcal{D}}$ not involutive, and the system controllable. □

4 Systems with drift

In the presence of drift, the reachable sets of the system (1) are no longer the orbits of $\mathcal{D}$. Consequently, Chow's theorem cannot tell us what the reachable sets are. The most it can tell us is that the reachable sets, which are contained in the orbits of $\mathcal{D}$, are contained in the integral manifold of $\Delta_{\mathcal{L}(\mathcal{D})}$. Thus the presence of drift significantly complicates the question of controllability. In particular, it is possible that the trajectories of a system cannot be restricted to a lower dimensional submanifold, and yet the system is not controllable, as the following example shows.

Example 4.1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^2, \\ \dot{x}_2 &= u. \end{aligned}$$

A little thought shows that the reachable set from a given $z = (z_1, z_2) \in \mathbb{R}^2$ is $\mathcal{R}(z) = \{w \in \mathbb{R}^2 : w_1 > z_1\} \cup \{z\}$. Thus the system is accessible, but not controllable. $\square$

For systems with drift, the following result from [4] represents the strongest assertion one can make about reachable sets in general.

Theorem 4.1. *Let $x \in \mathcal{M}$. Then $\mathcal{R}(x)$ is contained in the maximal integral manifold of $\Delta_{\mathcal{L}(\mathcal{D})}$ passing through x. Moreover, $\mathcal{R}(x)$ has a nonempty interior in this integral manifold.*

While the result above does not help us in determining controllability, it does allow us to determine if a system is accessible.

Corollary 4.1. *Suppose $\Delta_{\mathcal{L}(\mathcal{D})}(x)$ has dimension n for every $x \in \mathcal{M}$. Then the system is accessible.*

A comparison with Corollary 3.1 reveals that the same rank condition which guarantees controllability in the case of drift-free systems guarantees only accessibility in the presence of drift. To deduce controllability, additional conditions need to be checked. One such sufficient condition is described below.

The main impediment to controllability that arises in the presence of drift is that the orbits under $\mathcal{D}$ may not be the reachable sets. This is because, to reach points in the orbit, backward motion along a vector field in $\mathcal{D}$ is allowed, whereas points in the reachable set are those which can be reached by travelling only forward along the vector fields in $\mathcal{D}$. However, the reachable sets will equal the orbits if $\mathcal{D}$ had the property that $-X \in \mathcal{D}$ whenever $X \in \mathcal{D}$, because then one would be allowed to travel forward along both the vector fields X as well as $-X$, and travelling forward along $-X$ is the same as travelling backwards along X. $\mathcal{D}$ is *symmetric* if $-X \in \mathcal{D}$ for every $X \in \mathcal{D}$. Under symmetry of $\mathcal{D}$, the rank condition is sufficient to conclude controllability, as the following result from [6, Ch. 3] shows.

Theorem 4.2. *Suppose $\mathcal{D}$ is symmetric, and $\Delta_{\mathcal{D}}$ has dimension k at every $x \in \mathcal{M}$. Then the system (1) is controllable if and only if $k = n$.*

Since the inputs take positive as well as negative values, it is easy to see that $\mathcal{D}$ is symmetric if and only if $-F_0 \in \mathcal{D}$. A special case in which this happens is the drift-free case where $F_0 = 0$. Thus Theorem (4.2) yields Corollary (3.1) in the drift free case. More generally, $-F_0$ is contained in $\mathcal{D}$ if and only if the drift vector field F_0 is a linear combination (over $\mathbb{R}$) of the control vector fields.

Alternative sufficient conditions under which an accessible system is controllable are given in [7, 8]. These sufficient conditions are based on ideas involving recurrence.

The notion of accessibility that we have seen is weak in two senses. First, it places no limit on the time that might be required to reach states in the reachable set. Second, it leaves open the possibility that certain nearby states may be unreachable. Stronger notions of controllability are obtained by requiring additional properties that address these issues. Thus, *strong accessibility* [4] requires that the set of states reachable in any given period of time have a nonempty interior, *local controllability* [9] requires that the reachable set contain an open neighborhood of the initial point, and *small-time local controllability* [10] requires that the set of states reachable within any given period of time contain an open neighborhood of the initial point. All these notions of controllability are equivalent for a linear system.

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