# Controllability of Nonlinear Time-Varying Systems: Applications to Spacecraft Attitude Control Using Magnetic Actuation

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***Abstract*—Nonlinear controllability theory is applied to the time-varying attitude dynamics of a magnetically actuated spacecraft in a Keplerian orbit in the geomagnetic field. First, sufficient conditions for accessibility, strong accessibility and controllability of a general time-varying system are presented. These conditions involve application of Lie-algebraic rank conditions to the autonomous extended system obtained by augmenting the state of the original time-varying system by the time variable, and require the rank conditions to be checked only on the complement of a finite union of level sets of a finite number of smooth functions. At each point of each level set, it is sufficient to verify escape conditions involving Lie derivatives of the functions defining the level sets along linear combinations over smooth functions of vector fields in the accessibility algebra. These sufficient conditions are used to show that the attitude dynamics of a spacecraft actuated by three magnetic actuators and subjected to a general time-varying magnetic field are strongly accessible if the magnetic field and its time derivative are linearly independent at every instant. In addition, if the magnetic field is periodic in time, then the attitude dynamics of the spacecraft are controllable. These results are used to show that the attitude dynamics of a spacecraft actuated by three magnetic actuators in a closed Keplerian orbit in a nonrotating dipole approximation of the geomagnetic field are strongly accessible and controllable if the orbital plane does not coincide with the geomagnetic equatorial plane.**

***Index Terms*—Attitude control, controllability, magnetic actuation, time-varying systems.**

## I. Introduction

ATTITUDE control systems of earth satellites often utilize magnetic actuation, in which the mechanical torque required for attitude control is generated by the magnetic interaction between the geomagnetic field and on board electromagnets or *magnetic torquers*. In the past, magnetic torquers have mainly been used as secondary control actuators to assist other primary means of actuation for control and stabilization. Thus, magnetic torquers have been used for momentum management of reaction wheels, damping augmentation in gravity gradient stabilized spacecraft, and reorientation of the spin axis in spin-stabilized spacecraft. Because of the sharp reduction in the geomagnetic field intensity with increasing altitude, the mechanical torques produced by magnetic torquers are small in magnitude and, hence, magnetic torquers are not suitable as primary actuators in spacecraft that are large or operate at high altitudes. However, because of their low cost, weight and power requirements, magnetic torquers provide an attractive option for very small satellites operating at low altitudes. It is not surprising, therefore, that an increasing interest in small satellites has led to a recent growth of interest in attitude control systems that use only magnetic actuation; see, for instance, [1]–[6] and the references contained therein.

The principal difficulty involved in the use of magnetic actuation for attitude control is that the mechanical torque generated by any arrangement of magnetic torquers lies in a two-dimensional plane orthogonal to the local direction of the ambient magnetic field. Hence, it is natural to ask whether the three-dimensional rotational motion of a spacecraft is controllable using magnetic actuation. In the case where the inertial direction of the ambient magnetic field is constant, it is easy to see that the component of the spacecraft angular momentum along the direction of the magnetic field remains constant, and hence the attitude dynamics of the spacecraft are not controllable. However, an earth satellite in an inclined orbit in the geomagnetic field experiences a magnetic field that varies approximately periodically in time. While the time-varying nature of the magnetic field along an orbit introduces complications by making the attitude dynamics time varying, it also leads to the possibility of full attitude controllability. This possibility has been explored in recent work by several researchers. For instance, the authors of [1], [2], [4], and [7] have applied periodic linear quadratic optimal control to linearized models of the attitude dynamics by using periodic approximations for the time variation of the geomagnetic field along the orbit. Additional references can be found in [5]. The importance of the time variation of the magnetic field is highlighted by the results of [6], which show that the seemingly underactuated attitude dynamics of a magnetically actuated spacecraft that experiences a sufficiently rich orbital variation of the geomagnetic field can be almost globally asymptotically stabilized by continuous time-invariant state or attitude feedback. In contrast, the attitude dynamics of a spacecraft that is actuated by two independent actuators such as reaction thrusters cannot even be locally asymptotically stabilized by time-invariant static or dynamic feedback [8].

In this paper, we consider the controllability of the attitude dynamics of a magnetically actuated spacecraft translating along an orbit in the geomagnetic field. Previous work on attitude controllability includes [9], [10]. Reference [10] considered the controllability of the translational and rotational dynamics of a space-

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craft subjected to a gravity gradient torque in a central gravitational field. Reference [9] gives necessary and sufficient conditions for the controllability of the attitude dynamics of a spacecraft using body-fixed gas jet actuators yielding one, two, or three independent torques, and reaction wheels yielding three independent torques. The problem of controllability using magnetic actuation, however, is very different from the one considered in [9]. First, in the case of two gas jet actuators, the control torque is confined to a body-fixed two-dimensional plane. However, in the case of magnetic actuation, the control torque is confined to a two-dimensional plane that is determined by the inertial direction of the local geomagnetic field. Since the attitude dynamical equations for the angular velocity are usually written in terms of body-frame components, it follows that the angular velocity equation for a magnetically actuated spacecraft also involves the spacecraft attitude. Consequently, the results of [9] cannot be applied in the case of magnetic actuation. Second, the equations of motion of a magnetically actuated spacecraft are time varying due to the time variation of the geomagnetic field along the orbit. As a result, standard results on controllability of time-invariant systems do not apply directly.

Reference [11] used results on controllability of linear time-varying systems along with a fixed-point argument to obtain sufficient conditions for the local and global controllability of a time-varying input-affine system on $\mathbb{R}^n$ whose drift vector field can be expressed as a "linear" system with a time-varying, state-dependent state dynamics matrix. Motivated by the problem of rescuing a derelict spacecraft adrift in space, [12] and [13] considered the controllability of conservative periodic and almost periodic systems, respectively. Using techniques from topological dynamics, it was shown that a control system that has a compact state space manifold, and that is conservative and either periodic or almost periodic in the absence of control is globally controllable under the assumption of uniform controllability. Reference [14] showed that a similar result holds for a family of conservative uniformly almost periodic vector fields on a compact manifold if the assumption of uniform controllability is replaced by a more easily verifiable assumption involving a Lie-algebraic rank condition on the extended state space. In this paper, we give sufficient conditions for accessibility, strong accessibility and controllability of a time-varying nonlinear system using an approach that is closer in spirit to that of [9] and [15], and apply these conditions to the attitude dynamics of a magnetically actuated spacecraft.

In Section IV, we present the time-varying attitude dynamics and attitude kinematics of a magnetically actuated spacecraft subjected to a time-varying magnetic field. Because of the time-varying nature of the dynamics, standard results from nonlinear controllability theory are not directly applicable. Therefore, in Section III we consider the controllability of a general time-varying system. The original time-varying system is extended to a time-invariant system by augmenting the state with the time variable. We show that accessibility of the extended system implies accessibility of the original system. We state sufficient conditions for accessibility and strong accessibility of the original system, in terms of Lie-algebraic rank conditions (LARCs) involving the drift and control vector fields of the extended system. A convenient feature of the results we state is that they require the LARCs to be checked only on the complement of the union of level sets of a finite number of smooth functions instead of on the whole extended state–space. At points on the level set of each function, it is sufficient to check that the repeated Lie derivative of the function along some of the vector fields in the module generated by the accessibility algebra is nonzero, thereby verifying without computing higher order Lie brackets that it is possible to steer the trajectories of the system out of the level sets. Since Lie derivatives are easier to compute than Lie brackets, our results provide modifications of standard controllability results that are especially convenient to apply in practical applications where it often happens that the low-order Lie brackets computed initially satisfy the rank condition only on an open dense set whose complement is a union of level sets of smooth functions. We also state a sufficient condition for controllability of the original system, when the extended system is accessible. This condition, which requires the drift vector field of the original system to be time-invariant and weakly positively Poisson stable, and the control vector fields to be periodic with a common period, is related to results given in [9], [10], and [16] for time-invariant systems and results given in [12]–[14] for time-varying systems.

In Section IV, we apply the results of Section III to the attitude control of a spacecraft carrying three magnetic torquers in a general time-varying magnetic field. We show that if the magnetic field and its time derivative are linearly independent at every instant, then the attitude dynamics of the spacecraft are strongly accessible. In addition, if the magnetic field is periodic in time, then the attitude dynamics of the spacecraft are controllable. The controllability analysis presented in Section IV has two salient features. First, we verify the accessibility rank condition by computing Lie brackets of vector fields contained in the module generated by the accessibility algebra rather than vector fields contained in the accessibility algebra itself. This allows us the freedom of first simplifying terms in the vector fields before taking Lie brackets. Second, the vector fields that we compute satisfy the required rank conditions only on the complement of the union of level sets of a finite number of functions. While an application of standard controllability results would have necessitated computation of higher order Lie brackets, Theorem 3.1 of Section III allows us to conclude strong accessibility and controllability by simply computing the Lie derivatives of the functions defining the level sets along the vector fields already computed.

It should be noted that even though our results assume the presence of three magnetic torquers, the control system cannot be considered fully actuated because the resultant torque produced by the torquers is confined to the two-dimensional plane that is orthogonal to the instantaneous direction of the local magnetic field. Indeed, we provide an example in Section IV to show that, in general, the attitude dynamics of a spacecraft carrying only two magnetic torquers may not be accessible.

In Section V, we apply the results of Section IV to a spacecraft actuated by three magnetic torquers and moving along a closed Keplerian orbit in the geomagnetic field. Following [2], [3], and [7], we use a constant (nonrotating) dipole approximation for the geomagnetic field. We show that if the orbital plane of the spacecraft does not coincide with the geomagnetic equatorial plane, then the geomagnetic field and its time derivative along the orbital motion are linearly independent at every point along the orbit. Hence, the results of Section IV imply that the attitude dynamics of a spacecraft actuated by three magnetic torquers in the geomagnetic field are strongly accessible. The variation of the geomagnetic field along a closed orbit is periodic in time, and hence the controllability result from Section IV implies that the attitude dynamics of such a spacecraft are controllable.

We begin by reviewing the necessary mathematical preliminaries in Section II.

## II. Preliminaries

Let $\mathcal{M}$ be a $\mathrm{C}^\infty$ manifold. We will denote the set of $\mathrm{C}^\infty$ real-valued functions on $\mathcal{M}$ by $\mathrm{C}^\infty(\mathcal{M})$, and the set of $\mathrm{C}^\infty$ vector fields on $\mathcal{M}$ by $\mathcal{V}(\mathcal{M})$. Recall that $\mathrm{C}^\infty(\mathcal{M})$ is a ring under pointwise addition and multiplication. Given $\alpha$, $\beta \in \mathrm{C}^\infty(\mathcal{M})$, we denote the pointwise sum and pointwise product of $\alpha$ and $\beta$ by $\alpha+\beta$ and $\alpha\beta$, respectively. Given $\alpha \in \mathrm{C}^\infty(\mathcal{M})$ and $X$, $Y \in \mathcal{V}(\mathcal{M})$, we denote the pointwise product of $\alpha$ and $X$ by $\alpha X$, and the pointwise sum of $X$ and $Y$ by $X+Y$. $\mathcal{V}(\mathcal{M})$ is a module over $\mathrm{C}^\infty(\mathcal{M})$, that is, $\mathcal{V}(\mathcal{M})$ is closed under linear combinations formed with functions in $\mathrm{C}^\infty(\mathcal{M})$ as coefficients. A *submodule* of $\mathcal{V}(\mathcal{M})$ is a subset of $\mathcal{V}(\mathcal{M})$ that is a module over $\mathrm{C}^\infty(\mathcal{M})$.

A function $H : \mathcal{M} \to \mathbb{R}$ is *proper* if the inverse image of every compact subset of $\mathbb{R}$ under $H$ is compact.

Given $X \in \mathcal{V}(\mathcal{M})$, we let $\phi^X : (t,x) \mapsto \phi_t^X(x)$ denote the flow of $X$. The flow is defined on an open subset of $\mathbb{R} \times \mathcal{M}$ [17, Prop. 2.1.15]. The vector field $X$ is *complete* if its flow is defined on $\mathbb{R} \times \mathcal{M}$. If $X$ is complete, then $\phi_t^X : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism for every $t \in \mathbb{R}$.

A complete vector field $X \in \mathcal{V}(\mathcal{M})$ is *weakly positively Poisson stable* if, for every open set $\mathcal{U} \subseteq \mathcal{M}$ and every $t > 0$, there exists $T > t$ such that $\phi_T^X(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$.

Given $X \in \mathcal{V}(\mathcal{M})$ and $\gamma \in \mathrm{C}^\infty(\mathcal{M})$, the *Lie derivative* of $\gamma$ with respect to $X$ is the $\mathrm{C}^\infty$ function $L_X\gamma : \mathcal{M} \to \mathbb{R}$ given by $L_X\gamma(x) = (d/dt)|_{t=0}\gamma(\phi_t^X(x))$. If $Y \in \mathcal{V}(\mathcal{M})$, then we denote $L_Y L_X \gamma = L_Y(L_X\gamma)$. If $\alpha$, $\beta \in \mathrm{C}^\infty(\mathcal{M})$ and $Y \in \mathcal{V}(\mathcal{M})$, then

$$L_{\alpha X+\beta Y}\gamma = \alpha L_X\gamma + \beta L_Y\gamma. \tag{1}$$

Given vector fields $X, Y \in \mathcal{V}(\mathcal{M})$, their Lie bracket $[X,Y]$ is the unique vector field satisfying $L_{[X,Y]}\gamma = L_X L_Y\gamma - L_Y L_X\gamma$ for every $\gamma \in \mathrm{C}^\infty(\mathcal{M})$ [17, Sec. 2.2]. If $\beta \in \mathrm{C}^\infty(\mathcal{M})$, then

$$[X, \beta Y] = (L_X\beta)Y + \beta[X,Y]. \tag{2}$$

If $\mathcal{M}$ is an embedded submanifold of a manifold $\mathcal{N}$, and $\hat{X}$ and $\hat{Y}$ are $\mathrm{C}^\infty$ extensions to $\mathcal{N}$ of the vector fields $X$ and $Y$, respectively, then $[X,Y]$ is the restriction of $[\hat{X},\hat{Y}]$ to $\mathcal{M}$. In particular, if $\mathcal{N} = \mathbb{R}^n$ for some $n$, then, for every $x \in \mathcal{M}$, the canonical identification between $T_x\mathcal{N}$ and $\mathbb{R}^n$ yields

$$[X,Y](x) = \lim_{h\to 0}\frac{1}{h}\left[\hat{Y}\left(x+hX(x)\right) - \hat{X}\left(x+hY(x)\right)\right]. \tag{3}$$

The Lie bracket is a skew-symmetric, bilinear operator on $\mathcal{V}(\mathcal{M})$ that satisfies the Jacobi identity, thus making $\mathcal{V}(\mathcal{M})$ a Lie algebra. A *Lie subalgebra* of vector fields on $\mathcal{M}$ is a linear subspace over $\mathbb{R}$ of $\mathcal{V}(\mathcal{M})$ that is closed under the operation of Lie bracket.

Given a set $\mathcal{W} \subseteq \mathcal{V}(\mathcal{M})$ and $z \in \mathcal{M}$, we denote $\Delta_{\mathcal{W}}(z) = \mathrm{spn}\{X(z) : X \in \mathcal{W}\}$, where the span is over $\mathbb{R}$. The *distribution* generated by $\mathcal{W}$ is the union $\Delta_{\mathcal{W}} = \cup_{z\in\mathcal{M}}\Delta_{\mathcal{W}}(z)$.

In the sequel, we will require the notion of a Hamiltonian vector field on a symplectic manifold. Furthermore, we will use the well known fact that a symplectic manifold carries a natural volume form which is preserved by the flow of every Hamiltonian vector field defined on the manifold. We refer the reader to [17, Ch. 3] and [18, Ch. 8] for details.

## III. Controllability of Time-Varying Systems

Let $\mathcal{M}$ be an $m$-dimensional real analytic manifold. Consider the time-varying control system

$$\dot{y}(t) = f\left(y(t),t\right) + \sum_{i=1}^{q} g_i\left(y(t),t\right)u_i(t) \tag{4}$$

where the drift vector field $f$ and the control vector fields $g_1,\ldots,g_q$ are complete, real-analytic time-varying fields on the state space $\mathcal{M}$. Following [15], we assume that the control input vector $u = [u_1,\ldots,u_q]^{\mathrm{T}}$ is a piecewise continuous function of time that has finite right and left limits at every instant of discontinuity, and that takes values in a connected set $\Omega \subseteq \mathbb{R}^q$ containing 0 in its interior.

Given $x \in \mathcal{M}$, $h \in \mathbb{R}$ and $s \geq h$, the *reachable set* $R(x,h,s)$ of (4) from $(x,h)$ at $s$ is the set of all states that can be reached at time $s$ by following solutions of (4) that start at $x$ at time $h$. The reachable set $\mathcal{R}(x,h)$ of (4) from $x$ starting at $h$ is the set of all states that can be reached by following solutions of (4) that start at $x$ at time $h$, that is, $\mathcal{R}(x,h) = \cup_{s\geq h}R(x,h,s)$.

The system (4) is *accessible* at $x \in \mathcal{M}$ at time $h \in \mathbb{R}$ if $\mathcal{R}(x,h)$ has a nonempty interior in $\mathcal{M}$, and *strongly accessible* at $x \in \mathcal{M}$ at time $h \in \mathbb{R}$ if $R(x,h,s)$ has a nonempty interior in $\mathcal{M}$ for every $s > h$. The system (4) is accessible (strongly accessible) if it is accessible (strongly accessible) at every $x \in \mathcal{M}$ for every $h \in \mathbb{R}$. Finally, the system (4) is *controllable* if $\mathcal{R}(x,h) = \mathcal{M}$ for every $x \in \mathcal{M}$ and every $h \in \mathbb{R}$.

In order to study the accessibility and controllability properties of (4), it is convenient to introduce the *extended control system*

$$\dot{y}_{\mathrm{e}}(t) = F\left(y_{\mathrm{e}}(t)\right) + \sum_{i=1}^{q} G_i\left(y_{\mathrm{e}}(t)\right)u_i(t) \tag{5}$$

where the drift vector field $F$ and the control vector fields $G_1,\ldots,G_q$ given by $F(z) = (f(x),1)$ and $G_i(z) = (g_i(x,h),0)$, $i = 1,\ldots,q$, $z = (x,h) \in \mathcal{M}_{\mathrm{e}}$, are complete, real-analytic time-invariant fields on the $(m+1)$-dimensional real-analytic extended state–space manifold $\mathcal{M}_{\mathrm{e}} \overset{\mathrm{def}}{=} \mathcal{M} \times \mathbb{R}$. The extended system (5), which is autonomous, is obtained from the original time-varying system (4) by appending the original state with the time variable $t$ and augmenting the time-varying state–space equation (4) with the equation $\dot{t} = 1$.

Given $z \in \mathcal{M}_{\mathrm{e}}$ and $s \geq 0$, the *reachable set* $R_{\mathrm{e}}(z,s)$ of (5) from $z$ at $s$ is the set of all states in $\mathcal{M}_{\mathrm{e}}$ that can be reached in time $s$ by following solutions of (5) that start at $z$. The reachable set $\mathcal{R}_{\rm e}(z)$ of (5) from $z$ is the set of all states in $\mathcal{M}_{\rm e}$ that can be reached by following solutions of (5) that start at $z$, that is, $\mathcal{R}_{\rm e}(z)=\cup_{s\geq 0}R_{\rm e}(z,s)$. The extended system (5) is accessible if $\mathcal{R}_{\rm e}(z)$ has a nonempty interior in $\mathcal{M}_{\rm e}$ for every $z\in\mathcal{M}_{\rm e}$.

It is easy to see that $w \in \mathcal{R}_{\rm e}(z)$ implies that $\mathcal{R}_{\rm e}(w) \subseteq \mathcal{R}_{\rm e}(z)$ for every $z \in \mathcal{M}_{\rm e}$. It is also easy to see that $R(\pi_1(z),\pi_2(z),\pi_2(z)+s) = \pi_1(R_{\rm e}(z,s))$ and $\mathcal{R}(\pi_1(z),\pi_2(z))=\pi_1(\mathcal{R}_{\rm e}(z))$ for every $z\in\mathcal{M}_{\rm e}$ and $s\geq 0$, where $\pi_i:\mathcal{M}_{\rm e}\to\mathcal{M}$, $i=1,2$, are the projections given by $\pi_1((x,h))=x$, $\pi_2((x,h))=h$.

We denote by $\mathcal{D}$ the set of vector fields generated by the extended control system (5), that is, $\mathcal{D}$ is the set of vector fields on $\mathcal{M}_{\rm e}$ of the form $F+\mu_1G_1+\cdots+\mu_qG_q$, where $[\mu_1,\ldots,\mu_q]^{\rm T}\in\Omega$. The completeness of the drift and control vector fields of (5) implies that every vector field in $\mathcal{D}$ is complete. The *accessibility algebra* of (5) is the smallest subalgebra $\mathcal{A}$ of vector fields that contains $\mathcal{D}$. The accessibility algebra $\mathcal{A}$ generates a module $\mathcal{B}$ over the ring $\mathrm{C}^\infty(\mathcal{M}_{\rm e})$. Thus, $\mathcal{B}$ is the set of all linear combinations of vector fields in $\mathcal{A}$ with coefficients from $\mathrm{C}^\infty(\mathcal{M}_{\rm e})$, and is the smallest submodule of $\mathcal{V}(\mathcal{M}_{\rm e})$ that contains $\mathcal{A}$.

The *strong accessibility algebra* of (5) is the smallest subalgebra $\mathcal{A}_0$ of vector fields that contains the control vector fields $G_1,\ldots,G_q$ and is closed under the operation of Lie bracket with the drift vector field $F$. We denote by $\mathcal{B}_0$ the module generated by $\mathcal{A}_0$ over the ring $\mathrm{C}^\infty(\mathcal{M}_{\rm e})$. Equation (2) can be used to verify that $\mathcal{B}$ and $\mathcal{B}_0$ are Lie subalgebras. It follows that $\mathcal{B}$ is the smallest subalgebra of vector fields that contains $\mathcal{D}$ and that is a submodule of $\mathcal{V}(\mathcal{M}_{\rm e})$.

The distributions $\Delta_{\mathcal{A}}$ and $\Delta_{\mathcal{A}_0}$ are the *accessibility distribution* and the *strong accessibility distribution*, respectively, of the system (5). It is easy to see that, for every $z\in\mathcal{M}_{\rm e}$, $\Delta_{\mathcal{A}}(z)=\Delta_{\mathcal{B}}(z)$ and $\Delta_{\mathcal{A}_0}(z)=\Delta_{\mathcal{B}_0}(z)$. This fact allows us to calculate the ranks of the accessibility and strong accessibility distributions by computing vector fields in $\mathcal{B}$ and $\mathcal{B}_0$, respectively.

The set of vector fields $X\in\mathcal{V}(\mathcal{M}_{\rm e})$ satisfying $L_X\pi_2=0$ is a Lie subalgebra of $\mathcal{V}(\mathcal{M}_{\rm e})$ containing the vector fields $G_1,\ldots,G_r$. Moreover, since $L_F\pi_2$ has the constant value 1 on $\mathcal{M}_{\rm e}$, every vector field $X$ satisfying $L_X\pi_2=0$ satisfies $L_{[X,F]}\pi_2=L_XL_F\pi_2-L_FL_X\pi_2=0$. Thus, the Lie subalgebra of vector fields $X\in\mathcal{V}(\mathcal{M}_{\rm e})$ satisfying $L_X\pi_2=0$ is closed under the operation of Lie brackets with $F$, and hence contains the strong accessibility algebra $\mathcal{A}_0$. Therefore, it follows that, for every $z\in\mathcal{M}_{\rm e}$, the maximal integral manifold of $\mathcal{A}_0$ passing through $z$ is contained in the level set $\pi_2^{-1}(\pi_2(z))$ of $\pi_2$ containing $z$.

The following result, which is the main result of this section, gives sufficient conditions for accessibility, strong accessibility, and controllability of time-varying systems in terms of rank conditions on the accessibility and strong accessibility distributions of the extended system (5).

*Theorem 3.1:* Suppose there exist functions $\alpha_i:\mathcal{M}_{\rm e}\to\mathbb{R}$, $i=1,\ldots,r$, such that, for every $i=1,\ldots,r$, and every $z\in\alpha_i^{-1}(0)\setminus(\alpha_1^{-1}(0)\cup\cdots\cup\alpha_{i-1}^{-1}(0))$, there exist an integer $p$ and vector fields $X_1,\ldots,X_p$ in $\mathcal{B}$ such that $L_{X_1}\cdots L_{X_p}\alpha_i(z)\neq 0$. Then, the following statements hold.

i) If $\dim\Delta_{\mathcal{A}}(z)=m+1$ for every $z\in\mathcal{M}_{\rm e}$ satisfying $\alpha_i(z)\neq 0$, $i=1,\ldots,r$, then the time-varying system (4) as well as the extended system (5) are accessible.

ii) If $\dim\Delta_{\mathcal{A}_0}(z)=m$ for every $z\in\mathcal{M}_{\rm e}$ satisfying $\alpha_i(z)\neq 0$, $i=1,\ldots,r$, then (4) is strongly accessible.

iii) If the drift vector field $f$ is time invariant and Hamiltonian on $\mathcal{M}$ with a Hamiltonian function that is proper, the control vector fields $g_i$, $i=1,\ldots,q$, are periodic in time with a common period, and $\dim\Delta_{\mathcal{A}}(z)=m+1$ for every $z\in\mathcal{M}_{\rm e}$ satisfying $\alpha_i(z)\neq 0$, $i=1,\ldots,r$, then the system (4) is controllable.

The proof of Theorem 3.1 depends on the following lemma.

*Lemma 3.1:* Suppose $z\in\mathcal{M}_{\rm e}$, and let $\mathcal{U}\subseteq\mathcal{M}_{\rm e}$ be an open neighborhood of $z$. If $\alpha:\mathcal{U}\to\mathbb{R}$ is a $\mathrm{C}^\infty$ function such that $\alpha(w)=0$ for every $w\in\mathcal{R}_{\rm e}(z)\cap\mathcal{U}$, then, for every integer $p$ and for every choice of $p$ vector fields $X_1,\ldots,X_p$ in $\mathcal{B}$, it follows that $L_{X_1}\cdots L_{X_p}\alpha(w)=0$ for every $w\in\mathcal{R}_{\rm e}(z)\cap\mathcal{U}$.

*Proof:* Let $\mathcal{K}\subseteq\mathrm{C}^\infty(\mathcal{U})$ be the set of functions that are identically zero on $\mathcal{R}_{\rm e}(z)\cap\mathcal{U}$. Let $\Upsilon\subseteq\mathcal{V}(\mathcal{M}_{\rm e})$ be the set of vector fields $X$ having the property that $L_X\gamma\in\mathcal{K}$ whenever $\gamma\in\mathcal{K}$. It is easy to show from the definitions that $L_{X_1}\cdots L_{X_p}\gamma\in\mathcal{K}$ for every $X_1,\ldots,X_p\in\Upsilon$ and every $\gamma\in\mathcal{K}$.

It follows from (1) that $\Upsilon$ is a submodule of $\mathcal{V}(\mathcal{M}_{\rm e})$. Since $\mathcal{K}$ is a vector space under pointwise addition, for every $X_1$, $X_2\in\Upsilon$, $r_1$, $r_2\in\mathbb{R}$ and every $\gamma\in\mathcal{K}$, it follows that $L_{r_1X_1+r_2X_2}\gamma=r_1L_{X_1}\gamma+r_2L_{X_2}\gamma\in\mathcal{K}$ and $L_{[X_1,X_2]}\gamma=L_{X_1}L_{X_2}\gamma-L_{X_2}L_{X_1}\gamma\in\mathcal{K}$. Thus, $\Upsilon$ is a Lie subalgebra of $\mathcal{V}(\mathcal{M}_{\rm e})$.

We claim that $\Upsilon$ contains $\mathcal{D}$. To see this, consider $X\in\mathcal{D}$, $\gamma\in\mathcal{K}$, and $w\in\mathcal{R}_{\rm e}(z)\cap\mathcal{U}$. There exists $\epsilon>0$ such that $\phi_t^X(w)\in\mathcal{U}$ for every $t\in[0,\epsilon)$. Since $X\in\mathcal{D}$, it follows that $\phi_t^X(w)\in\mathcal{R}_{\rm e}(w)\cap\mathcal{U}\subseteq\mathcal{R}_{\rm e}(z)\cap\mathcal{U}$ for every $t\in[0,\epsilon)$. Hence, $\gamma(\phi_t^X(w))=0$ for every $t\in[0,\epsilon)$. Consequently, $L_X\gamma(w)=0$. Since $w\in\mathcal{R}_{\rm e}(z)\cap\mathcal{U}$ was chosen arbitrarily, it follows that $L_X\gamma(w)=0$ for every $w\in\mathcal{R}_{\rm e}(z)\cap\mathcal{U}$. Since $\gamma\in\mathcal{K}$ and $X\in\mathcal{D}$ were chosen arbitrarily, it follows that $\mathcal{D}\subseteq\Upsilon$. Thus, $\Upsilon$ is a subalgebra containing $\mathcal{D}$ and, hence, contains $\mathcal{A}$. Since $\Upsilon$ is a submodule of $\mathcal{V}(\mathcal{M}_{\rm e})$, it follows that $\mathcal{B}\subseteq\Upsilon$. The result now follows by noting that if $\alpha\in\mathcal{K}$, then, for every integer $p$ and for every choice of $p$ vector fields $X_1,\ldots,X_p$ from $\mathcal{B}\subseteq\Upsilon$, $L_{X_1}\cdots L_{X_p}\alpha\in\mathcal{K}$. ■

*Proof of Theorem 3.1:* Denote $\mathcal{Q}_1=\alpha_1^{-1}(0)$. For each $i=2,\ldots,r$, denote $\mathcal{Q}_i=\alpha_i^{-1}(0)\setminus(\mathcal{Q}_1\cup\cdots\cup\mathcal{Q}_{i-1})$. Finally, let $\mathcal{Q}_{r+1}=\mathcal{M}_{\rm e}\setminus(\mathcal{Q}_1\cup\cdots\cup\mathcal{Q}_r)$, so that $\mathcal{M}_{\rm e}$ is the disjoint union of the sets $\mathcal{Q}_1,\ldots,\mathcal{Q}_{r+1}$.

Consider $z_0=(x,h)\in\mathcal{M}_{\rm e}$. Then, $z_0\in\mathcal{Q}_i$ for some $i=1,\ldots,r+1$, and $\alpha_j(z_0)\neq 0$ for every $j=1,\ldots,i-1$. There exists an open neighborhood $\mathcal{U}_0$ of $z_0$ such that $\alpha_j(z)\neq 0$ for every $j=1,\ldots,i-1$ and every $z\in\mathcal{U}_0$. First, consider the case where $i<r+1$. If $\mathcal{R}_{\rm e}(z_0)\cap\mathcal{U}_0\subseteq\alpha_i^{-1}(0)$, then Lemma 3.1 implies that $L_{X_1}\cdots L_{X_p}\alpha_i(z_0)=0$ for every choice of $X_1,\ldots,X_p$ in $\mathcal{B}$. However, this contradicts the hypotheses of Theorem 3.1. Hence, we conclude that $\mathcal{R}_{\rm e}(z_0)\cap\mathcal{U}_0$ contains a point $z_1$ outside $\mathcal{Q}_i$. By our construction of $\mathcal{U}_0$, $z_1\notin\mathcal{Q}_j$ for $j=1,\ldots,i-1$. Hence $z_1\in\mathcal{Q}_{i+1}\cup\cdots\cup\mathcal{Q}_{r+1}$. Applying the same arguments as before, it can be shown that $\mathcal{R}_{\rm e}(z_1)$ contains a point $z_2\in\mathcal{Q}_{i+2}\cup\cdots\cup\mathcal{Q}_{r+1}$. Continuing in this way, we can construct a finite sequence of points $z_1,\ldots,z_{r-i+1}$ such that $z_j\in\mathcal{Q}_{i+j}\cup\cdots\cup\mathcal{Q}_{r+1}$ and $z_{j+1}\in\mathcal{R}_{\rm e}(z_j)$ for every $j=1,\ldots,r-i$, and $z_{r-i+1}\in\mathcal{Q}_{r+1}$ so that $\mathcal{R}_{\rm e}(z_{r-i+1})\subseteq$

$\cdots \subseteq \mathcal{R}_{\rm e}(z_1) \subseteq \mathcal{R}_{\rm e}(z_0)$. In the case where $i = r+1$, $z_{r-i+1} = z_0 \in \mathcal{Q}_{r+1}$ trivially.

i) Assume $\dim \Delta_{\mathcal{A}}(z) = m+1$ for every $z \in \mathcal{Q}_{r+1}$. Then, by [15, Cor. 4.6], [19, Th. 2.2], and [20, Th. 3.9], it follows that $\mathcal{R}_{\rm e}(z_{r-i+1})$ has a nonempty interior in $\mathcal{M}_{\rm e}$. However, $\mathcal{R}_{\rm e}(z_{r-i+1}) \subseteq \mathcal{R}_{\rm e}(z_0)$. Hence it follows that $\mathcal{R}_{\rm e}(z_0)$ has a nonempty interior in $\mathcal{M}_{\rm e}$. Since projections map open sets to open sets, it follows that $\mathcal{R}(x,h) = \pi_1(\mathcal{R}_{\rm e}(z_0))$ has a nonempty interior in $\mathcal{M}$. Since $z_0 = (x,h)$ was chosen arbitrarily, it follows that the system (5) as well as the system (4) are accessible.

ii) Next, assume that $\dim \Delta_{\mathcal{A}_0}(z) = m$ for every $z \in \mathcal{Q}_{r+1}$. There exist $T_1, \ldots, T_{r-i+1} \geq 0$ such that $z_i \in R_{\rm e}(z_{i-1}, T_i)$ for every $i = 1, \ldots, r-i+1$. Thus $z_{r-i+1} \in R_{\rm e}(z_0, T)$, where $T = T_1 + \cdots + T_{r-i+1}$. Since $\mathcal{A}_0$ is a subalgebra of analytic vector fields, the strong accessibility distribution $\Delta_{\mathcal{A}_0}$ has the maximal integral manifolds property [21, Cor. 2.1.7], [22]. Let $\mathcal{I}$ denote the maximal integral manifold of $\mathcal{A}_0$ through $z_0$. Let $X \in \mathcal{D}$ and $s > 0$. Since $z_{r-i+1} \in R_{\rm e}(z_0, T)$, it follows from [15, Lemma 3.5] that $\phi_T^X(\mathcal{I})$ is a maximal integral manifold of $\mathcal{A}_0$ through $z_{r-i+1}$. Since $\dim \Delta_{\mathcal{A}_0}(z_{r-i+1}) = m$, $\phi_T^X(\mathcal{I})$ and hence $\phi_s^X(\mathcal{I})$ is an $m$-dimensional submanifold of $\mathcal{M}_{\rm e}$ [22, Th. 4.1]. Moreover, $\phi_s^X(\mathcal{I})$ is contained in $\phi_s^X(\pi_2^{-1}(\pi_2(z_0))) = \pi_2^{-1}(s+h) = \mathcal{M} \times \{s+h\}$. Since $\mathcal{M} \times \{s+h\}$ is an $m$-dimensional submanifold of $\mathcal{M}_{\rm e}$, it follows that $\phi_s^X(\mathcal{I})$ is an open subset of $\mathcal{M} \times \{s+h\}$. Since [15, Th. 4.5] implies that $R_{\rm e}(z_0, s)$ has a nonempty interior in $\phi_s^X(\mathcal{I})$, it follows that $R_{\rm e}(z_0, s)$ has a nonempty interior in $\mathcal{M} \times \{s+h\}$. Since the projection $\pi_1$ is a diffeomorphism between $\mathcal{M}$ and $\mathcal{M} \times \{s+h\}$, it follows that $R(x, h, h+s) = \pi_1(R_{\rm e}(z_0, s))$ has a nonempty interior in $\mathcal{M}$. Since $z_0 = (x,h)$ and $s > 0$ were chosen arbitrarily, it follows that the system (4) is strongly accessible.

iii) Suppose the drift vector field $f$ is time-invariant and Hamiltonian on $\mathcal{M}$ with a proper Hamiltonian function $H : \mathcal{M} \to \mathbb{R}$, and the control vector fields $g_i$, $i = 1, \ldots, q$, are periodic in time with a common period $T > 0$. Our assumption implies that the vector field $F$ is time invariant and the vector fields $G_i$, $i = 1, \ldots, q$ on $\mathcal{M}_{\rm e}$ are periodic in the second argument with period $T$. Let $\widehat{\mathcal{M}} \overset{\rm def}{=} \mathcal{M} \times {\rm S}^1$ and define the map $\rho : \mathcal{M}_{\rm e} \to \widehat{\mathcal{M}}$ by $\rho(x,h) = (x, e^{\imath 2\pi h/T})$, where we identify ${\rm S}^1$ with the unit circle in the complex plane, and $\imath = \sqrt{-1}$. Letting $\hat{\pi}_1 : \widehat{\mathcal{M}} \to \mathcal{M}$ denote the projection on the first factor, it follows that $\hat{\pi}_1 \circ \rho = \pi_1$.Since the vector fields $F$ and $G_i$, $i = 1, \ldots, q$, are periodic in the second argument with period $T$, these vector fields "project" onto well-defined vector fields $\hat{F}$ and $\hat{G}_i$, $i = 1, \ldots, q$, on $\widehat{M}$ satisfying $\rho \circ \phi_s^X = \phi_s^{\hat{X}} \circ \rho$ for every $s \in \mathbb{R}$ and $X \in \{F, G_1, \ldots, G_q\}$. The function $\rho$ maps the reachable sets of the system (5) to the reachable sets of the system

$$\dot{\hat{y}}(t) = \hat{F}(\hat{y}(t)) + \sum_{i=1}^{q} \hat{G}_i(\hat{y}(t)) u_i(t) \tag{6}$$

on $\widehat{\mathcal{M}}$.

Under the assumptions on $f$, Liouville's Theorem [18, Sec. 38] implies that the flow of $f$ preserves volume on $\mathcal{M}$. Since the flow of $\hat{F}$ is given by $\phi_h^{\hat{F}}(\hat{z}) = (\phi_h^f(x), e^{\imath 2\pi h/T} w)$ for every $\hat{z} = (x,w) \in \widehat{\mathcal{M}} \times {\rm S}^1$, it follows that the flow of $\hat{F}$ also preserves volume on $\widehat{\mathcal{M}}$. $\widehat{\mathcal{M}}$ can be written as the union of sets of the form $H^{-1}([a,b]) \times {\rm S}^1$, $b > a$, each of which is compact by properness. Since the Hamiltonian function is constant along the flow of $f$, each set of the form $H^{-1}([a,b]) \times {\rm S}^1$, $b > a$, is positively invariant under the flow of $\hat{F}$. Poincaré's Recurrence Theorem [18, Sec. 16] now implies that the vector field $\hat{F}$ is weakly positively Poisson stable on each set of the form $H^{-1}([a,b]) \times {\rm S}^1$, $b > a$, and hence on $\widehat{\mathcal{M}}$ (see also [10, Th. 2]).

Now, suppose $\dim \Delta_{\mathcal{A}}(z) = m+1$ for every $z \in \mathcal{M}_{\rm e}$ satisfying $\alpha_i(z) \neq 0$, $i = 1, \ldots, r$. Then, point i) of Theorem 3.1 implies that the system (5) is accessible on $\mathcal{M}_{\rm e}$. Since $\rho$ is a local diffeomorphism that maps the reachable sets of (5) to those of (6), it follows that the system (6) is accessible on $\widehat{\mathcal{M}}$. Theorem 3 of [10] now implies that the system (6) is controllable on $\widehat{\mathcal{M}}$.

Consider $z = (x,h) \in \mathcal{M}_{\rm e}$, and let $\hat{z} = \rho(z)$. Letting $\widehat{\mathcal{R}}(\hat{z})$ denote the reachable set of the system (6) from $\hat{z}$, it follows from the controllability of (6) that $\widehat{\mathcal{M}} = \widehat{\mathcal{R}}(\hat{z}) = \rho(\mathcal{R}_{\rm e}(z))$. Now $\mathcal{R}(x,h) = \pi_1(\mathcal{R}_{\rm e}(z)) = \hat{\pi}_1(\rho(\mathcal{R}_{\rm e}(z))) = \hat{\pi}_1(\widehat{\mathcal{R}}(\hat{z}))= \hat{\pi}_1(\widehat{\mathcal{M}}) = \mathcal{M}$. Since $(x,h) \in \mathcal{M}_{\rm e}$ was chosen arbitrarily, it follows that the system (4) is controllable on $\mathcal{M}$. ∎

*Remark 3.1:* It is well known that in the case of analytic systems, accessibility implies that the accessibility rank condition in i) of Theorem 3.1 holds everywhere. See, for instance, [15, Cor. 4.6]. Hence, from a theoretical point of view, Theorem 3.1 is equivalent to standard results on (global) controllability of nonlinear systems which require verifying rank conditions on the accessibility and strong accessibility distributions at every state. However, applications of these standard results may require computing as many Lie brackets as necessary to verify the rank conditions at every point, while, in practice, only a small number of Lie brackets can be conveniently computed. It often happens in practical applications of controllability theory that the low order Lie brackets which are computed initially satisfy the rank conditions only on an open dense set having a nonempty complement. Typically, the complement is a union of level sets of smooth functions. While an application of standard results would require a tedious computation of additional higher order Lie brackets, Theorem 3.1 simply requires showing the existence of nonzero Lie derivatives of the functions defining the level sets along vector fields in the submodule generated by the Lie brackets already computed. The existence of nonzero Lie derivatives verifies "escape" conditions guaranteeing that the trajectories of the system can be steered away from the level sets where the rank condition has not been verified. Since Lie derivatives of scalar functions are easier to compute than Lie brackets of vector fields, Theorem 3.1 provides an extension of standard controllability results that proves more convenient in applications. The convenience provided by Theorem 3.1 is illustrated by our application of Theorem 3.1 to the problem of attitude controllability of a magnetically actuated spacecraft in the next section.

*Remark 3.2:* Statement i) of Theorem 3.1 asserts the accessibility of the autonomous extension of a time-varying system. However, the proof of the statement does not rely on the special structure of the extended system or of the extended state space. Hence, i) of Theorem 3.1 applies to any general autonomous system. A similar remark applies to Lemma 3.1.

## IV. Attitude Controllability in a Time-Varying Magnetic Field

In this section, we apply the results of the previous section to the attitude dynamics of a magnetically actuated spacecraft subjected to a time-varying external magnetic field. We begin by giving the equations of motion governing the attitude dynamics of a rigid spacecraft carrying magnetic actuators in a time-varying magnetic field.

We describe the attitude of the spacecraft using a matrix $R \in \mathrm{SO}(3)$ such that the multiplication of the body components of a vector by $R$ gives the components of that vector with respect to a reference inertial frame. The attitude kinematics of the spacecraft are then described by the equation

$$\dot{R}(t) = R(t)\,(\omega(t)\times) \tag{7}$$

where $\omega(t) \in \mathbb{R}^3$ denotes the instantaneous body-frame components of the angular velocity of the spacecraft relative to the reference inertial frame, and, for every $a \in \mathbb{R}^3$

$$(a\times) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{8}$$

is the matrix representation of the linear map $b \mapsto (a \times b)$ with $b \in \mathbb{R}^3$ and $\times$ denoting the familiar operation of cross product on $\mathbb{R}^3$.

The attitude dynamics of the spacecraft are governed by Euler's equation

$$J\dot{\omega}(t) = -\left(\omega(t) \times J\omega(t)\right) + \tau(t) \tag{9}$$

where $J \in \mathbb{R}^{3\times 3}$ is the symmetric positive–definite moment-of-inertia matrix of the spacecraft with respect to its body frame, and $\tau(t) \in \mathbb{R}^3$ is the vector of instantaneous body-frame components of the external torque acting on the spacecraft.

In the case of magnetic actuation, the external torque is the result of the magnetic interaction between on-board magnetic torquers and an external magnetic field. Letting $B(t) \in \mathbb{R}^3$ denote the vector of instantaneous components of the external magnetic field with respect to the reference inertial frame, the instantaneous body-frame components of the resultant external torque due to the interaction between $q$ magnetic torquers and the external magnetic field is given by

$$\tau(t) = \sum_{i=1}^{q} u_i(t)\left[k_i \times R^{\mathrm{T}}(t)B(t)\right] \tag{10}$$

where, for every $i = 1, \ldots, q$, $u_i(t) \in \mathbb{R}$ is the magnitude of the instantaneous magnetic dipole moment of the $i$th magnetic torquer, and $k_i$ is the vector of body-frame components of the unit vector along the dipole moment of the $i$th magnetic torquer. In this paper, we consider a spacecraft carrying three magnetic torquers that generate linearly independent magnetic dipole moment vectors, that is, we take $q = 3$ and assume $k_1$, $k_2$ and $k_3$ to be linearly independent. In this case, the equations (7), (9) and (10) describing the attitude motion give rise to the time-varying control system

$$\dot{y}(t) = f\left(y(t)\right) + \sum_{i=1}^{3} g_i\left(y(t), t\right) u_i(t) \tag{11}$$

on the 6-dimensional manifold $\mathcal{M} \stackrel{\mathrm{def}}{=} \mathrm{SO}(3) \times \mathbb{R}^3$, with the time-invariant drift vector field $f$ and the time-varying control vector fields $g_i$, $i = 1, 2, 3$, given by

$$f(R,\omega) = \left(R(\omega\times), -J^{-1}(\omega \times J\omega)\right) \tag{12}$$

$$g_i(R,\omega,t) = \left(0, -J^{-1}\left(R^{\mathrm{T}}B(t) \times k_i\right)\right). \tag{13}$$

We assume the time variation of the magnetic field to be real analytic and bounded. Then, the drift vector fields $g_1$, $g_2$ and $g_3$ are real analytic and complete. The drift vector field $f$, which is clearly analytic, represents the torque-free dynamics of a rigid body, and is known to be complete.

The time-varying control system (11) gives rise to the time-invariant extended control system

$$\dot{y}_{\mathrm{e}}(t) = F\left(y_{\mathrm{e}}(t)\right) + \sum_{i=1}^{3} G_i\left(y_{\mathrm{e}}(t)\right) u_i(t) \tag{14}$$

on the seven-dimensional extended state space $\mathcal{M}_{\mathrm{e}} \stackrel{\mathrm{def}}{=} \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}$.

In this section, we apply Theorem 3.1 to the vector fields of the control system (14) to deduce the controllability of the system (11). For this purpose, we treat the vector fields $F$, $G_1$ and $G_2$ as vector fields on $\mathbb{R}^{3\times 3} \times \mathbb{R}^3 \times \mathbb{R}$ and use (3) to compute several Lie brackets involving the vector fields $F$, $G_1$ and $G_2$. An alternative approach to computing Lie brackets on vector bundles over $\mathrm{SO}(3)$ is described in [10] and [23].

We begin with the Lie brackets

$$\begin{aligned} Z_1(z) &\stackrel{\mathrm{def}}{=} [G_1, F](z) \\ &= \Big(-R(b_1\times), J^{-1}\Big\{(b_1 \times J\omega) + b \times (\omega \times k_1) \\ &\qquad + \Big(R^{\mathrm{T}}\dot{B}(t) \times k_1\Big)\Big\}, 0\Big) \end{aligned} \tag{15}$$

$$\begin{aligned} Z_2(z) &\stackrel{\mathrm{def}}{=} [G_2, F](z) \\ &= \Big(-R(b_2\times), J^{-1}\Big\{(b_2 \times J\omega) + b \times (\omega \times k_2) \\ &\qquad + \Big(R^{\mathrm{T}}\dot{B}(t) \times k_2\Big)\Big\}, 0\Big) \end{aligned} \tag{16}$$

where $z = (R, \omega, t) \in \mathcal{M}_{\mathrm{e}}$, $b = R^{\mathrm{T}}B(t)$, $b_1 = J^{-1}(b \times k_1)$ and $b_2 = J^{-1}(b \times k_2)$. Strictly speaking, $b$, $b_1$ and $b_2$ are $\mathbb{R}^3$-valued functions on $\mathcal{M}_{\mathrm{e}}$. However, we will suppress the arguments of these functions for notational convenience.

Define $\gamma_1 : \mathcal{M}_{\mathrm{e}} \to \mathbb{R}$ by $\gamma_1(z) = b^{\mathrm{T}}(z)(k_1 \times k_2)$ and $\delta_i^j : \mathcal{M}_{\mathrm{e}} \to \mathbb{R}$, $i, j = 1,2$, by $\delta_1^j(z) = -(k_2 \times b)^{\mathrm{T}}(\omega \times k_j)$ and $\delta_2^j(z) = (k_1 \times b)^{\mathrm{T}}(\omega \times k_j)$, $j = 1, 2$, where $z = (R, \omega, t) \in$

$\mathcal{M}_{\rm e}$. It can be verified by direct substitution that, $\gamma_1(z)(b\times(\omega\times k_j)) + \delta_1^j(z)(b\times k_1) + \delta_2^j(z)(b\times k_2) = 0$, $j=1,2$, for every $z\in\mathcal{M}_{\rm e}$. For every $z\in\mathcal{M}_{\rm e}$, let

$$X_3(z) \stackrel{\rm def}{=} \Big(-R(b_1\times), J^{-1}\Big\{(b_1\times J\omega) + \Big(R^{\rm T}\dot{B}(t)\times k_1\Big)\Big\}, 0\Big) \quad (17)$$

$$X_4(z) \stackrel{\rm def}{=} \Big(-R(b_2\times), J^{-1}\Big\{(b_2\times J\omega) + \Big(R^{\rm T}\dot{B}(t)\times k_2\Big)\Big\}, 0\Big) \quad (18)$$

and define $Z_3 \stackrel{\rm def}{=} \gamma_1 X_3$ and $Z_4 \stackrel{\rm def}{=} \gamma_1 X_4$. Then $Z_3=\gamma_1 Z_1 - \delta_1^1 G_1 - \delta_2^1 G_2$ and $Z_4 = \gamma_1 Z_2 - \delta_1^2 G_1 - \delta_2^2 G_2$. Since the vector fields $Z_1$ and $Z_2$ are contained in the strong accessibility algebra $\mathcal{A}_0$ of the system (14), it follows that $Z_3, Z_4\in\mathcal{B}_0$.

Next, we compute

$$\begin{aligned} X_5(z) &\stackrel{\rm def}{=} [X_3,X_4](z) \\ &= \Big(-R(b_3\times), J^{-1}\Big\{b_3\times J\omega + \Big(R^{\rm T}\dot{B}(t)\Big) \times((b_2\times k_1)-(b_1\times k_2))\Big\}, 0\Big) \end{aligned} \quad (19)$$

where $b_3 = J^{-1}\{(b_1\times b)\times k_2 - (b_2\times b)\times k_1\} - (b_1\times b_2)$, and define $Z_5 \stackrel{\rm def}{=} \gamma_1^2 X_5$. Then $Z_5 = [Z_3,Z_4] - (L_{X_3}\gamma_1)Z_4 + (L_{X_4}\gamma_1)Z_3$, and thus $Z_5\in\mathcal{B}_0$.

Define functions $\beta_i:\mathcal{M}_{\rm e}\to\mathbb{R}$, $i=1,\ldots,4$, by $\beta_1(z) \stackrel{\rm def}{=} -(b\times k_2)^{\rm T}(Jb_3\times J\omega)$, $\beta_2(z) \stackrel{\rm def}{=} (b\times k_1)^{\rm T}(Jb_3\times J\omega)$, $\beta_3(z) \stackrel{\rm def}{=} -b^{\rm T}J\omega$ and $\beta_4(z) \stackrel{\rm def}{=} \gamma_1(z)(b^{\rm T}Jb_3)$ for every $z=(R,\omega,t)\in\mathcal{M}_{\rm e}$. It can be verified by direct substitution and algebraic manipulation that the smooth functions $\beta_i$, $i=1,\ldots,4$, satisfy

$$\beta_1(z)b_1(z) + \beta_2(z)b_2(z) + \gamma_1(z)\beta_3(z)b_3(z) + \beta_4(z)\omega = 0 \quad (20)$$

for every $z=(R,\omega,t)\in\mathcal{M}_{\rm e}$. Define $X_6 \stackrel{\rm def}{=} \beta_1 X_3 + \beta_2 X_4 + \gamma_1\beta_3 X_5 - \beta_4 F$, $X_7 \stackrel{\rm def}{=} [X_6,G_1]$, $X_8 \stackrel{\rm def}{=} [X_6,G_2]$, $Z_6 \stackrel{\rm def}{=} \gamma_1 X_6 = \beta_1 Z_3 + \beta_2 Z_4 + \beta_3 Z_5 - \beta_4\gamma_1 F$, $Z_7 \stackrel{\rm def}{=} \gamma_1 X_7$ and $Z_8 = \gamma_1 X_8$. Since $L_{G_i}\gamma_1 = 0$ for $i=1,2$, it follows that $Z_7 = [Z_6,G_1]$ and $Z_8=[Z_6,G_2]$. On noting that $L_{G_i}\beta_4 = 0$ for $i=1,2$, it follows from (2) that $Z_7$ and $Z_8$ are linear combinations over $\mathrm{C}^\infty(\mathcal{M}_{\rm e})$ of the vector fields $Z_3$, $Z_4$, $Z_5$, $[Z_3,G_i]$, $[Z_4,G_i]$, $[Z_5,G_i]$, and $[F,G_i]$, $i=1,2$, each of which is in $\mathcal{B}_0$. Hence, it follows that $Z_7, Z_8\in\mathcal{B}_0$.

To obtain expressions for $X_7$ and $X_8$, we use (20) to write $X_6 = \beta_1 Y_1 + \beta_2 Y_2 + \gamma_1\beta_3 Y_3 + \beta_4 Y_4$, where, for every $z=(R,\omega,t)\in\mathcal{M}_{\rm e}$, $Y_1(z) = (0, J^{-1}(R^{\rm T}\dot{B}(t)\times k_1), 0)$, $Y_2(z) = (0,J^{-1}(R^{\rm T}\dot{B}(t)\times k_2),0)$, $Y_3(z) = (0,J^{-1}\{R^{\rm T}\dot{B}(t)\times(b_2\times k_1 - b_1\times k_2)\},0)$, and $Y_4(z) = (0,0,-1)$. Direct computation shows that $L_{G_1}\beta_1 = L_{G_2}\beta_2 = \beta_4$, $L_{G_i}\beta_j = 0$ for every $j=1,2,3$, $i=1,2$, $i\neq j$, and $[Y_4,G_i] = Y_i$ for $i=1,2$, while $[Y_j,G_i]=0$ for every $j=1,2,3$ and $i=1,2$. Hence, it follows that

$$\begin{aligned} X_7(z) &= \beta_4(z)Y_1(z) + \beta_4(z)[Y_4,G_1](z) \\ &= \Big(0, 2\beta_4(z)J^{-1}\Big(R^{\rm T}\dot{B}(t)\times k_1\Big), 0\Big) \end{aligned} \quad (21)$$

$$\begin{aligned} X_8(z) &= \beta_4(z)Y_2(z) + \beta_4(z)[Y_4,G_2](z) \\ &= \Big(0, 2\beta_4(z)J^{-1}\Big(R^{\rm T}\dot{B}(t)\times k_2\Big), 0\Big) \end{aligned} \quad (22)$$

for every $z=(R,\omega,t)\in\mathcal{M}_{\rm e}$.

In the proof of the theorem that we present later, we will require an explicit computation of the quantity $(b_1\times b_2)^{\rm T}b_3$ and the function $\beta_4$. To perform the necessary computations, we recall that if $p,\ q\in\mathbb{R}^3$ and $A\in\mathbb{R}^{3\times3}$ is a symmetric invertible matrix, then $pq^{\rm T} - qp^{\rm T} = (q\times p)\times$, $Ap\times Aq = (\det A)A^{-1}(p\times q)$, and $A(p\times) + (p\times)A = \{(\mathrm{trace}\, A)p - Ap\}\times$. The second identity shown previously follows from properties of the scalar triple product on $\mathbb{R}^3$, while the third follows from an identity given in [23]. Using these identities, the expressions for $(b_1\times b_2)^{\rm T}b_3$ and $\beta_4$ can be simplified to yield

$$\begin{aligned} &(\det J)^2(b_1\times b_2)^{\rm T}b_3 \\ &=(\det J)\beta_4(z) \\ &=\big(b^{\rm T}(k_1\times k_2)\big)^2\big[(b^{\rm T}Jb)(\mathrm{trace}\ J) - 2(Jb)^{\rm T}Jb\big]. \end{aligned} \quad (23)$$

Our next two results give sufficient conditions for strong accessibility and controllability of (11). Based on the vector fields computed previously, we show that the accessibility and strong accessibility distributions satisfy the required rank conditions on the complement of a finite union of level sets of functions. While an application of standard sufficient conditions for global accessibility and strong accessibility would require computation of additional vector fields in the accessibility and strong accessibility algebras, Theorem 3.1 allows us to deduce accessibility and strong accessibility after merely computing Lie derivatives of the functions defining the level sets along some of the vector fields computed previously.

*Theorem 4.1:* Suppose the magnetic field satisfies

$$B(t)\times\dot{B}(t)\neq 0, \qquad t\in\mathbb{R}. \quad (24)$$

Then, the attitude dynamics described by (11) are strongly accessible on $\mathcal{M} = \mathrm{SO}(3)\times\mathbb{R}^3$.

*Proof:* We begin by recalling the function $\gamma_1:\mathcal{M}_{\rm e}\to\mathbb{R}$ and introducing the function $\gamma_2:\mathcal{M}_{\rm e}\to\mathbb{R}$ given by $\gamma_1(z) = b^{\rm T}(z)(k_1\times k_2)$, $\gamma_2(z) = b^{\rm T}(z)p(z)$, where $p:\mathcal{M}_{\rm e}\to\mathbb{R}^3$ is the function given by $p(z) = (\mathrm{trace}\ J)Jb(z) - 2J^2b(z)$.

Consider $z=(R,\omega,t)\in\mathcal{M}_{\rm e}$ satisfying $\gamma_1(z)\neq 0$ and $\gamma_2(z)\neq 0$. Note that $\gamma_1(z)\neq 0$ implies that $b(z)\notin\mathrm{span}\{k_1,k_2\}$. Equation (23) implies that the vectors $b_1$, $b_2$, and $b_3$ are linearly independent. Also, if $(R^{\rm T}\dot{B}(t))^{\rm T}(b\times k_1) = (R^{\rm T}\dot{B}(t))^{\rm T}(b\times k_2) = 0$, then the vector $R^{\rm T}\dot{B}(t)$ is parallel to the vector $(b\times k_1)\times(b\times k_2) = [(b\times k_1)^{\rm T}k_2]b = \gamma_1(z)R^{\rm T}B(t)$, which contradicts (24). Hence, at least one of $(R^{\rm T}\dot{B}(t))^{\rm T}(b\times k_1)$ and $(R^{\rm T}\dot{B}(t))^{\rm T}(b\times k_2)$ is nonzero.

The facts that $b(z)\notin\mathrm{span}\{k_1,k_2\}$ and $b_1$, $b_2$, $b_3$ are linearly independent can be used to show that, if $(R^{\rm T}\dot{B}(t))^{\rm T}(b\times k_1)\neq 0$, then the vectors $G_1(z)$, $G_2(z)$, $X_3(z)$, $X_4(z)$, $X_5(z)$ and $X_7(z)$, and hence the vectors $G_1(z)$, $G_2(z)$, $Z_3(z)$, $Z_4(z)$, $Z_5(z)$ and $Z_7(z)$ are linearly independent. Similarly, if $(R^{\rm T}\dot{B}(t))^{\rm T}(b\times k_2)\neq 0$, then the vectors $G_1(z)$, $G_2(z)$, $Z_3(z)$, $Z_4(z)$, $Z_5(z)$ and $Z_8(z)$ are linearly independent.

It now follows that, for every $z \in \mathcal{M}_{\rm e}$ satisfying $\gamma_1(z) \neq 0 \neq \gamma_2(z)$, there exist six vector fields in $\mathcal{B}_0$ that are linearly independent at $z$, that is, $\dim \Delta_{\mathcal{A}_0}(z) = \dim \Delta_{\mathcal{B}_0}(z) = 6$.

We note that the vector fields $Z_3$, $Z_4$, $Z_5$, $Z_7$ and $Z_8$ are all zero on $\gamma_1^{-1}(0)$. Consequently, the rank of the strong accessibility distribution at points in $\gamma_1^{-1}(0)$ cannot be ascertained from the vector fields considered so far. Instead of computing higher order Lie brackets to ascertain the rank, we will next compute Lie derivatives of the functions $\gamma_1$ and $\gamma_2$, and show that the escape conditions of Theorem 3.1 are satisfied at points on the level sets of $\gamma_1$ and $\gamma_2$.

Consider $z = (R, \omega, t) \in \mathcal{M}_{\rm e}$ such that $\gamma_1(z) = (k_1 \times k_2)^{\rm T} R^{\rm T} B(t) = 0$. It is easy to compute $L_F \gamma_1(z) = (k_1 \times k_2)^{\rm T}[(b \times \omega) + R^{\rm T}\dot{B}]$ and $L_{G_i} L_F \gamma_1(z) = -(k_1 \times k_2)^{\rm T}[b \times J^{-1}(b \times k_i)] = -[(k_1 \times k_2) \times b]^{\rm T} J^{-1}(b \times k_i)$, $i = 1,2,3$. The vectors $J^{-1}(b \times k_i)$, $i = 1,2,3$, are orthogonal to the vector $Jb$, and two of them are linearly independent since $k_1$, $k_2$ and $k_3$ are linearly independent. Now, if $L_{G_i} L_F \gamma_1(z) = 0$ for every $i = 1,2,3$, then the vectors $J^{-1}(b \times k_i)$, $i = 1,2,3$, are also orthogonal to the vector $(k_1 \times k_2) \times b$. Thus, $L_{G_i} L_F \gamma_1(z) = 0$ for every $i = 1,2,3$ implies that the vectors $Jb$ and $(k_1 \times k_2) \times b$ are parallel, which implies that $b^{\rm T} J b = 0$. However, this contradicts the fact that $J$ is a positive-definite matrix. Hence we conclude that $L_{G_i} L_F \gamma_1(z) \neq 0$ for some $i = 1,2,3$.

The Lie derivatives of the function $\gamma_2$ along the vector fields $X_3$ and $X_4$ are given by $L_{X_3}\gamma_2(z) = 2(b_1 \times b)^{\rm T} p(z)$ and $L_{X_4}\gamma_2(z) = 2(b_2 \times b)^{\rm T} p(z)$.

Consider $z = (R, \omega, t) \in \mathcal{M}_{\rm e}$ such that $p(z) = 0$. Then $z$ satisfies

$$Jb(z) = \frac{\operatorname{trace} J}{2} b(z). \tag{25}$$

Equation (25) implies that $b(z)$ is along a principal axis of inertia, and the corresponding principal moment of inertia is $\lambda_1 = (1/2)\operatorname{trace} J$. Letting $\lambda_2$ and $\lambda_3$ denote the other two principal moments of inertia so that $\operatorname{trace} J = \lambda_1 + \lambda_2 + \lambda_3$, it follows from $\lambda_1 = (\operatorname{trace} J)/2$ that $\lambda_1 = \lambda_2 + \lambda_3$. Since $\lambda_2$ and $\lambda_3$ are positive, it follows that $\lambda_1$ is an unrepeated eigenvalue of $J$. Let $v_1 \in \mathbb{R}^3$ and $v_2 \in \mathbb{R}^3$ be linearly independent vectors orthogonal to the eigenvector of $J$ corresponding to the eigenvalue $\lambda_1$. Define $\gamma_3 : \mathcal{M}_{\rm e} \to \mathbb{R}$ and $\gamma_4 : \mathcal{M}_{\rm e} \to \mathbb{R}$ by $\gamma_3(w) \stackrel{\rm def}{=} v_1^{\rm T} b(w)$, $\gamma_4(w) \stackrel{\rm def}{=} v_2^{\rm T} b(w)$, $w \in \mathcal{M}_{\rm e}$. It is clear from the above discussion that $\gamma_3(z) = \gamma_4(z) = 0$ for every $z \in \mathcal{M}_{\rm e}$ satisfying $p(z) = 0$. For every $z = (R, \omega, t) \in \mathcal{M}_{\rm e}$, an easy computation yields $L_{X_3}\gamma_3(z) = (b_1 \times b)^{\rm T} v_1$ and $L_{X_4}\gamma_3(z) = (b_2 \times b)^{\rm T} v_1$. Now, consider $z = (R, \omega, t) \in \mathcal{M}_{\rm e}$ such that $\gamma_1(z) \neq 0$ and $\gamma_3(z) = 0$. Then, $L_{X_3}\gamma_3(z) = L_{X_4}\gamma_3(z) = \gamma_3(z) = 0$ implies that the vectors $b$, $b_1 \times b$ and $b_2 \times b$ are linearly dependent so that $b^{\rm T}[(b_1 \times b) \times (b_2 \times b)] = 0$. However, it can be shown after algebraic manipulation that

$$b^{\rm T}\left[(b_1 \times b) \times (b_2 \times b)\right] = (\det J^{-1})(b^{\rm T} b)(b^{\rm T} J b)\gamma_1(z) \tag{26}$$

which is nonzero. It follows that at least one of $L_{X_3}\gamma_3(z)$ and $L_{X_4}\gamma_3(z)$ is nonzero. Consequently, at least one of $L_{Z_3}\gamma_3(z) = \gamma_1(z) L_{X_3}\gamma_3(z)$ and $L_{Z_4}\gamma_3(z) = \gamma_1(z) L_{X_4}\gamma_3(z)$ is nonzero. Identical arguments can be used to show that at least one of the Lie derivatives $L_{Z_3}\gamma_4(z)$ and $L_{Z_4}\gamma_4(z)$ is nonzero for every $z \in \mathcal{M}_{\rm e}$ such that $\gamma_1(z) \neq 0$ and $\gamma_4(z) = 0$.

Next, we consider $z = (R, \omega, t) \in \mathcal{M}_{\rm e}$ such that $\gamma_1(z) \neq 0$, $\gamma_3(z) \neq 0$, $\gamma_4(z) \neq 0$ and $\gamma_2(z) = 0$. Since $\gamma_3(z) \neq 0 \neq \gamma_4(z)$, it follows that $p(z) \neq 0$. If $L_{X_3}\gamma_2(z) = 0$ and $L_{X_4}\gamma_2(z) = 0$, then the expressions for $L_{X_3}\gamma_2(z)$ and $L_{X_4}\gamma_2(z)$ along with $\gamma_2(z) = 0$ imply that $b^{\rm T}[(b_1 \times b) \times (b_2 \times b)] = 0$, which contradicts (26). It follows that at least one of $L_{X_3}\gamma_2(z)$ and $L_{X_4}\gamma_2(z)$ and, hence, at least one of $L_{Z_3}\gamma_2(z)$ and $L_{Z_4}\gamma_2(z)$ is nonzero. Thus, we have shown the following.

i) For every $z \in \mathcal{M}_{\rm e}$ such that $\gamma_1(z) \neq 0$ and $\gamma_2(z) \neq 0$, it follows that $\dim \Delta_{\mathcal{A}_0}(z) = 6$.

ii) For every $z \in \mathcal{M}_{\rm e}$ such that $\gamma_1(z) = 0$, at least one of the Lie derivatives $L_{G_i} L_F \gamma_1(z)$, $i = 1,2,3$, is nonzero.

iii) For every $z \in \mathcal{M}_{\rm e}$ and every $i = 3,4$, such that $\gamma_1(z) \neq 0$ and $\gamma_i(z) = 0$, at least one of the Lie derivatives $L_{Z_3}\gamma_i(z)$ and $L_{Z_4}\gamma_i(z)$ is nonzero.

iv) For every $z \in \mathcal{M}_{\rm e}$ such that $\gamma_1(z) \neq 0$, $\gamma_3(z) \neq 0$, $\gamma_4(z) \neq 0$ and $\gamma_2(z) = 0$, at least one of the Lie derivatives $L_{Z_3}\gamma_2(z)$ and $L_{Z_4}\gamma_2(z)$ is nonzero.

Hence applying ii) of Theorem 3.1 with $\alpha_1 = \gamma_1$, $\alpha_2 = \gamma_3$, $\alpha_3 = \gamma_4$, and $\alpha_4 = \gamma_2$, it follows that the attitude dynamics described by (11) are strongly accessible on $\mathcal{M}$. ■

It is interesting to note that the attitude dynamics are not accessible if (24) is violated on $\mathbb{R}$. Indeed, if $B(t) \times \dot{B}(t) = 0$ for all $t \in \mathbb{R}$, then the unit vector along the magnetic field is a constant vector $p \in \mathbb{R}^3$. Since all inertial torques are orthogonal to $p$, the angular momentum of the spacecraft along $p$ is constant. In other words, the function $\alpha : \mathcal{M} \to \mathbb{R}$ defined by $\alpha(x) = \omega^{\rm T} J R^{\rm T} p$, $x = (R, \omega) \in \mathcal{M}$, is constant along the solutions of (11). The level sets of $\alpha$ have an empty interior in $\mathcal{M}$. Consequently, the attitude dynamics of the spacecraft are not accessible.

*Remark 4.1:* It can easily be shown that (24) implies that the matrix

$$\overline{\Gamma}_0 \stackrel{\rm def}{=} \lim_{T \to \infty} \frac{1}{T} \int_0^T (B(t)\times)(B(t)\times)^{\rm T}\, dt$$

considered in [6] is positive definite. Thus, our sufficient condition (24) for strong accessibility implies the sufficient condition of [6] for stabilizability. The sufficient condition of [6] is weaker, because the matrix $\overline{\Gamma}_0$ is positive definite if and only if (24) holds on an open (and possibly strict) subset of $[0, \infty)$.

*Theorem 4.2:* Suppose the magnetic field satisfies (24) and, in addition, is periodic. Then, the attitude dynamics described by (11) are controllable on $\mathcal{M} = \mathrm{SO}(3) \times \mathbb{R}^3$.

*Proof:* Consider the functions $\gamma_i$, $i = 1,2,3,4$, introduced in the proof of Theorem 4.1. In the proof of Theorem 4.1, it was shown that six of the seven vector fields $G_1$, $G_2$, $Z_3$, $Z_4$, $Z_5$, $Z_7$ and $Z_8$ from $\mathcal{B}_0 \subseteq \mathcal{B}$ are linearly independent at every point $z \in \mathcal{M}_{\rm e}$ satisfying $\gamma_1(z) \neq 0 \neq \gamma_2(z)$. Since the third component of each of the vector fields listed before is zero while the third component of the vector field $F \in \mathcal{B}$ is 1, it follows that, for every $z \in \mathcal{M}_{\rm e}$ satisfying $\gamma_1(z) \neq 0 \neq \gamma_2(z)$, seven of the eight vector fields $F$, $G_1$, $G_2$, $Z_3$, $Z_4$, $Z_5$, $Z_7$, and $Z_8$ from $\mathcal{B}$ are linearly independent at $z$, that is, $\dim \Delta_{\mathcal{A}}(z) = \dim \Delta_{\mathcal{B}}(z) = 7$. It was also shown in the proof of Theorem 4.1 that, for every $z \in \mathcal{M}_{\rm e}$ such that $\gamma_1(z) = 0$, at least one of the

Lie derivatives $L_{G_i}L_F\gamma_1(z)$, $i=1,2,3$, is nonzero, for every $z\in\mathcal{M}_{\rm e}$ such that $\gamma_1(z)\neq 0$ and $\gamma_3(z)=0$ ($\gamma_4(z)=0$), at least one of the Lie derivatives $L_{Z_3}\gamma_3(z)$ ($L_{Z_3}\gamma_4(z)$) and $L_{Z_4}\gamma_3(z)$ ($L_{Z_4}\gamma_4(z)$) is nonzero while, for every $z\in\mathcal{M}_{\rm e}$ such that $\gamma_2(z)=0$, $\gamma_1(z)\neq 0$, $\gamma_3(z)\neq 0$, and $\gamma_4(z)\neq 0$, at least one of the Lie derivatives $L_{Z_3}\gamma_2(z)$ and $L_{Z_4}\gamma_2(z)$ is nonzero.

Now suppose the magnetic field is periodic in time. Then it follows that the control vector fields $g_1$, $g_2$ and $g_3$ in (11) are periodic in time with a common period. The time-invariant drift vector field in (11) represents the torque-free motion of a rigid body, and is known to be Hamiltonian on $\mathcal{M}=\mathrm{SO}(3)\times\mathbb{R}^3$ with the Hamiltonian $H(R,\omega)=(1/2)\omega^{\rm T}J\omega$ equal to the rotational kinetic energy of the rigid body. Since the kinetic energy is positive-definite and quadratic in the angular velocity, and SO(3) is compact, it follows that the Hamiltonian function $H$ is proper. It now follows from iii) of Theorem 3.1 that the system (11) is controllable on $\mathcal{M}=\mathrm{SO}(3)\times\mathbb{R}^3$. ■

Theorem 4.1 holds for a spacecraft carrying three magnetic torquers acting along three linearly independent axes. However, the Lie brackets used to verify the accessibility rank conditions in the proof of Theorem 4.1 involved only two of the three control vector fields. The third control vector field was only used to verify one of the escape conditions. This might suggest that accessibility may hold even in the case of a spacecraft carrying two magnetic torquers. Our example below demonstrates that the attitude dynamics of a spacecraft carrying two magnetic torquers may fail to be accessible even if (24) holds.

*Example 4.1:* Consider a spacecraft carrying two magnetic torquers in a time-varying magnetic field that satisfies (24) but is confined to lie in a two-dimensional stationary plane. Such a magnetic field would be experienced, for instance, by a spacecraft moving along an orbit in the magnetic field of a nonrotating dipole whose axis lies in the orbital plane. The attitude dynamics of such a spacecraft are described by (14) on $\mathcal{M}_{\rm e}=\mathrm{SO}(3)\times\mathbb{R}^3\times\mathbb{R}$ with $u_3=0$. Moreover, there exists a constant vector $p\in\mathbb{R}^3$ such that $p^{\rm T}B(t)=0$ for all $t\geq 0$.

Let $v_1, v_2, v_3\in\mathbb{R}^3$ denote unit vectors along the body-fixed principal axes of inertia of the spacecraft. The vectors $v_1$, $v_2$ and $v_3$ are mutually orthogonal eigenvectors of the moment-of-inertia matrix $J$ of the spacecraft. Assume that the unit vectors $k_1$ and $k_2$ defining the axes of the two magnetic torquers lie in the principal plane of inertia formed by $v_1$ and $v_2$.

Define functions $\alpha_i:\mathcal{M}_{\rm e}\to\mathbb{R}$ and $\gamma_i:\mathcal{M}_{\rm e}\to\mathbb{R}$ by $\alpha_i(z)=p^{\rm T}Rv_i$ and $\gamma_i(z)=\omega^{\rm T}v_i$, where $i=1,2$ and $z=(R,\omega,t)\in\mathcal{M}_{\rm e}$. Consider $z=(R,\omega,t)\in\mathcal{M}_{\rm e}$. The equalities $\alpha_i(z)=0$, $i=1,2$, imply that the principal plane of inertia formed by vectors $v_1$ and $v_2$ coincides with the plane in which the magnetic field is confined, that is, the principal axis along $v_3$ is parallel to the vector $p$. The equalities $\gamma_i(z)=0$, $i=1,2$, imply that the angular velocity vector $\omega$ coincides with the principal axis along $v_3$. In particular, $\gamma_i(z)=0, i=1,2$ implies that the vectors $\omega$ and $J\omega$ are parallel. Thus, $\mathcal{Q}_{\rm e}\stackrel{\rm def}{=}\{z\in\mathcal{M}_{\rm e}:\alpha_i(z)=\gamma_i(z)=0, i=1,2\}$ is the set of all extended states in $\mathcal{M}_{\rm e}$ in which the spacecraft is rotating about the principal axis of inertia along $v_3$ with the principal plane of inertia containing the axes of the magnetic torquers coinciding with the plane to which the magnetic field is confined.

It can be shown that the functions $\alpha_i$ and $\gamma_i$, $i=1,2$, have linearly independent differentials at every $z\in\mathcal{Q}_{\rm e}$. Hence, it follows that $\mathcal{Q}_{\rm e}$ is a three-dimensional submanifold of $\mathcal{M}_{\rm e}$. Moreover, the tangent space to $\mathcal{Q}_{\rm e}$ at every $z\in\mathcal{Q}_{\rm e}$ consists of tangent vectors to $\mathcal{M}_{\rm e}$ along which the differentials at $z$ of each $\alpha_i$ and $\gamma_i$, $i=1,2$, are zero. Since the functions $\alpha_i,\gamma_i$, $i=1,2$, are independent of time, the manifold $\mathcal{Q}_{\rm e}$ is of the form $\mathcal{Q}_{\rm e}=\mathcal{Q}\times\mathbb{R}$, where $\mathcal{Q}$ is a two-dimensional submanifold of $\mathcal{M}$.

Next, consider $z=(R,\omega,t)\in\mathcal{Q}_{\rm e}$. On noting that the vectors $R^{\rm T}p$, $\omega$, $J\omega$ and $v_3$ are all parallel, it follows that $L_F\alpha_i(z)=p^{\rm T}R(\omega\times v_i)=0$ and $L_F\gamma_i(z)=-v_1^{\rm T}J^{-1}(\omega\times J\omega)=0$ for $i=1,2$. The Lie derivatives $L_{G_j}\alpha_i(z)$ are clearly zero for $i$, $j=1,2$. On noting that the vectors $v_i$, $J^{-1}v_i$, $k_i$, $i=1,2$, and $R^{\rm T}B(t)$ are all contained in the same two-dimensional plane, it follows that $L_{G_j}\gamma_i(z)=-v_i^{\rm T}J^{-1}(R^{\rm T}B(t)\times k_j)=0$ for $i$, $j=1,2$. Since $z\in\mathcal{Q}_{\rm e}$ was chosen arbitrarily, it follows that the vector fields $F$, $G_1$ and $G_2$ are tangent to the submanifold $\mathcal{Q}_{\rm e}$. It now follows that every vector field in $\mathcal{A}$ is also tangent to the submanifold $\mathcal{Q}_{\rm e}$, that is, for every $z\in\mathcal{Q}_{\rm e}$, $\Delta_{\mathcal{A}}(z)$ is contained in the tangent space to $\mathcal{Q}_{\rm e}$ at $z$. Consequently, the maximal integral manifold of $\mathcal{A}$ containing $z\in\mathcal{Q}_{\rm e}$ is contained in $\mathcal{Q}_{\rm e}$. Given $h\geq 0$ and $x=(R,\omega)\in\mathcal{M}$ such that $z=(x,h)\in\mathcal{Q}_{\rm e}$, the reachable set $\mathcal{R}(x,h)=\pi_1(\mathcal{R}_{\rm e}(z))\subseteq\pi_1(\mathcal{Q}_{\rm e})=\mathcal{Q}$ which has an empty interior in $\mathcal{M}$. Thus the attitude dynamics of the spacecraft are not accessible at states contained in $\mathcal{Q}_{\rm e}$.

## V. Attitude Controllability in the Geomagnetic Field

In the previous section, we have shown that the attitude dynamics of a spacecraft carrying three independent magnetic torquers in a magnetic field are strongly accessible and controllable if the ambient magnetic field and its time derivative are linearly independent at every instant, and the magnetic field varies periodically in time. In this section, we show that the time variation of a constant dipole approximation of the geomagnetic field along a closed Keplerian orbit satisfies the conditions mentioned previously if the orbital plane does not coincide with the geomagnetic equatorial plane. Therefore, the attitude dynamics of a spacecraft moving along such a closed Keplerian orbit in the geomagnetic field are strongly accessible and controllable.

At low altitudes, the geomagnetic field can be approximated by the magnetic field of a dipole. Assuming the geomagnetic field to be generated by a magnetic dipole of dipole moment $\mathbf{m}\in\mathbb{R}^3$ that coincides with earth's axis of rotation, the vector $B$ of inertial components of the geomagnetic field at a point having the geocentric position vector $\mathbf{r}\in\mathbb{R}^3$ is given by

$$B(\mathbf{r})=\frac{\mu}{r^5}\left[3(\mathbf{m}^{\rm T}\mathbf{r})\mathbf{r}-r^2\mathbf{m}\right] \tag{27}$$

where $r=\sqrt{\mathbf{r}^{\rm T}\mathbf{r}}$ and $\mu$ is the dipole strength [24, p. 783].

Consider a spacecraft moving along a closed Keplerian orbit in the magnetic field (27). The geocentric position vector of such a spacecraft satisfies $\mathbf{r}(t)\times\dot{\mathbf{r}}(t)=\mathbf{h}$ for all $t$, where $\mathbf{h}$ is the constant specific angular momentum of the spacecraft [25, Ch. 2]. We will assume that $\mathbf{h}\neq\mathbf{0}$ since $\mathbf{h}=\mathbf{0}$ corresponds to degenerate straight line orbits which have no practical utility. The orbit lies in a stationary two-dimensional plane that is orthogonal to $\mathbf{h}$. The orbital plane contains the geomagnetic poles if and only if $\mathbf{m}^{\mathrm{T}}\mathbf{h}=0$, while the orbital plane coincides with the geomagnetic equatorial plane if and only if $\mathbf{m}^{\mathrm{T}}\mathbf{r}(t)=\mathbf{m}^{\mathrm{T}}\dot{\mathbf{r}}(t)=0$ for every $t$.

Differentiating (27) along the translational motion of the spacecraft yields

$$\begin{aligned}\dot{B}(t) &\stackrel{\text{def}}{=} \frac{d}{dt}B\left(\mathbf{r}(t)\right)\\ &= -5\frac{\dot{r}}{r}B(t)+\frac{\mu}{r^5}\\ &\quad\times\left[3(\mathbf{m}^{\mathrm{T}}\dot{\mathbf{r}})\mathbf{r}-2(\mathbf{r}^{\mathrm{T}}\dot{\mathbf{r}})\mathbf{m}+3(\mathbf{m}^{\mathrm{T}}\mathbf{r})\dot{\mathbf{r}}\right] \quad (28)\end{aligned}$$

where we have suppressed the time dependence of $\mathbf{r}$ for convenience and denoted $B(t)\stackrel{\text{def}}{=}B(\mathbf{r}(t))$. After some simplification, (27) and (28) yield

$$\begin{aligned}B(t)\times\dot{B}(t)=\frac{3\mu^2}{r^{10}}&\left[\left\{(\mathbf{r}\times\mathbf{m})^{\mathrm{T}}\mathbf{h}\right\}(\mathbf{r}\times\mathbf{m})\right.\\ &\left.+2(\mathbf{m}^{\mathrm{T}}\mathbf{r})^2\mathbf{h}+(\mathbf{m}^{\mathrm{T}}\mathbf{r})(\mathbf{m}^{\mathrm{T}}\mathbf{h})\mathbf{r}\right]. \quad (29)\end{aligned}$$

First, assume that the spacecraft orbital plane contains neither the geomagnetic equator nor the geomagnetic poles. The second assumption implies that the three vectors $\mathbf{r}\times\mathbf{m}$, $\mathbf{h}$ and $\mathbf{r}$ appearing in (29) are linearly independent at every point in the orbit. The first assumption implies that if $\mathbf{m}^{\mathrm{T}}\mathbf{r}=\mathbf{0}$ at a point in the orbit, then, at the same point, the coefficient $(\mathbf{r}\times\mathbf{m})^{\mathrm{T}}\mathbf{h}$ of the vector $\mathbf{r}\times\mathbf{m}$ in (29) equals $r^2\mathbf{m}^{\mathrm{T}}\dot{\mathbf{r}}\neq 0$. Thus, at every point on the orbit, at least one of the coefficients of the linearly independent vectors $\mathbf{r}\times\mathbf{m}$, $\mathbf{h}$ and $\mathbf{r}$ in (29) is nonzero. It follows that (24) holds at every point along the orbit.

Next, assume that the spacecraft orbital plane contains the geomagnetic poles. Then the vector $\mathbf{r}\times\mathbf{m}$ is orthogonal to the orbital plane and hence parallel to the specific angular momentum $\mathbf{h}$. Therefore, $\mathbf{r}\times\mathbf{m}=(\mathbf{h}^{\mathrm{T}}\mathbf{h})^{-1}\{(\mathbf{r}\times\mathbf{m})^{\mathrm{T}}\mathbf{h}\}\mathbf{h}$. Moreover, $\mathbf{m}^{\mathrm{T}}\mathbf{h}=0$. Thus, (29) reduces to

$$B(t)\times\dot{B}(t)=\frac{3\mu^2}{r^{10}}\left[(\mathbf{h}^{\mathrm{T}}\mathbf{h})^{-1}\left\{(\mathbf{r}\times\mathbf{m})^{\mathrm{T}}\mathbf{h}\right\}^2+2(\mathbf{m}^{\mathrm{T}}\mathbf{r})^2\right]\mathbf{h}. \quad (30)$$

The quantity $(\mathbf{r}\times\mathbf{m})^{\mathrm{T}}\mathbf{h}$ is zero only if $\mathbf{r}$ is parallel to the geomagnetic polar axis, while the quantity $\mathbf{m}^{\mathrm{T}}\mathbf{r}$ is zero only if $\mathbf{r}$ lies in the geomagnetic equatorial plane. Since $\mathbf{h}\neq\mathbf{0}$, it follows from (30) that (24) holds at every point along the orbit.

The previous arguments show that (24) holds along every orbit that does not lie in the geomagnetic equatorial plane. In addition, the variation of the geomagnetic field (27) along a closed orbit is clearly periodic with period equal to the orbital period. Theorems 4.1 and 4.2 now lead to the following result.

*Theorem 5.1:* Assume that the geomagnetic field is given by (27). Then the attitude dynamics of a spacecraft in a closed Keplerian orbit that does not lie in the geomagnetic equatorial plane are strongly accessible and controllable.

*Remark 5.1:* Equation (27) gives an axisymmetric, nonrotating approximation for the geomagnetic field. In reality, the geomagnetic field involves higher order terms that are not axisymmetric [24]. In addition, the polar axis of the geomagnetic field does not coincide with the axis of rotation of the earth. These facts imply that any point fixed in a geocentric nonrotating frame experiences a periodic variation in the geomagnetic field due to the rotation of the earth. Thus, in general, a spacecraft in a closed Keplerian orbit would experience an almost periodic (but not necessarily periodic) magnetic field variation resulting from the periodic orbital motion of the satellite and the periodic rotation of the geomagnetic field. The analysis of [14] suggests that iii) of Theorem 3.1 and, hence, Theorem 4.2 extends to the almost periodic case. However, the analysis used in this section to show that (24) holds does not extend directly to more realistic models of the geomagnetic field. Thus the controllability of the attitude dynamics of a spacecraft in a Keplerian orbit under a more realistic model of the geomagnetic field remains open.

## VI. CONCLUSION

We have given sufficient conditions for accessibility, strong accessibility, and controllability of a time-varying control system in terms of the drift and control vector fields of the corresponding time-invariant extended system. Our conditions require the rank conditions to be checked only on the complement of a finite union of level sets of a finite number of smooth functions. At each point of each level set, it is sufficient to verify escape conditions involving Lie derivatives of the functions defining the level sets along linear combinations over smooth functions of vector fields in the accessibility algebra. These conditions have been used to show that the attitude dynamics of a spacecraft actuated by three magnetic torquers and subjected to a time-varying magnetic field are strongly accessible if the magnetic field and its time derivative are linearly independent at every instant, and controllable if, in addition, the magnetic field is periodic in time. The time variation of a constant dipole approximation of the geomagnetic field along a closed Keplerian orbit has been shown to possess both these properties in the case where the orbital plane does not coincide with the geomagnetic equatorial plane.

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