Numerical Solutions for Ideal Magnetohydrodynamics

An Introduction

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Outline of Presentation

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Introduction

- **Magnetohydrodynamics**: combination of fluid dynamics and electromagnetics → Dynamics of an ionized gas ("plasma") in the presence of magnetic fields
- **Applications**: astrophysics, hypersonic vehicles, thermonuclear fusion, plasma propulsion drag reduction, stealth
- **Wave structure**: magneto-acoustic waves + Alfven waves
- **Ideal Magnetohydrodynamics**: obtained by neglecting dissipative mechanisms (viscosity, thermal and electric conductivity, etc)
- **Hyperbolic** (Non-strict), Non-convex conservative system of equations
- **Numerical Solutions**: Straightforward application of algorithms for Euler equations not possible
Plasma Dynamics

- **Ionized Gas**: ionization occurs at high temperatures due to high-energy collisions; comprises of positive ions and electrons
  - Dynamic process: simultaneous ionization and de-ionization; fraction of ionized gas depends on temperature
- Ionization also possible through **Photo-Ionization** and **Electric Discharge**
  - Photo-ionization: Ionization caused by subjecting gas to UV / X / Gamma rays
  - Electric discharge: ionization caused by presence of very strong electric fields
- **Plasma**: conducting fluid satisfying the plasma criterion
  - **Macroscopic Neutrality** - charge separation allowed for distances of the order of **Debye length**
  - Large number of charged particles inside the...
Plasma Dynamics

- **Debye Shielding Length**: Maximum distance over which the Coulombic field of a particle can be felt; important physical parameter
- **“Fourth State of Matter”**: solid, liquid, gas, plasma; no sharp temperature for gas → plasma phase transition
- **Collective Effects**: Plasma properties dependent on particle interactions and collective effects
  - Each charged particle interacts simultaneously with large number of particles through electromagnetic fields
  - Each particle is a source of electro-static and magnetic fields; aside from externally applied electromagnetic fields
- **Natural Occurrence of Plasma**: Solar corona, Solar wind, Earth’s Ionosphere (Van Allen Radiation Belts)
Earth’s Magnetosphere and the Solar Wind

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Artist Rendition of Solar Wind
Created by: K. Endo

Photo Courtesy of Prof. Yohsuke Kamide
National Geophysical Data Center
Tokamak - Plasma Containment

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The Magnetohydrodynamic Approximation

- **Single-fluid continuum assumption** → Plasma treated as a single conducting fluid (may be composed of many species)
- **Extension of the Navier-Stokes** to include electromagnetic force and energy terms
  - Mass Conservation
    \[
    \frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{u}) = 0 \tag{1}
    \]
  - Momentum Conservation
    \[
    \frac{\partial (\rho \mathbf{u})}{\partial t} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B} - \nabla . (\rho \mathbf{I} + \frac{1}{2} \rho \mathbf{uu}) + \psi \tag{2}
    \]
  - Energy Conservation
    \[
    \frac{1}{2} \rho \frac{Dv^2}{Dt} + \rho \frac{De}{Dt} = -\rho \nabla . \mathbf{u} + \mathbf{E} . \mathbf{J} + \phi \tag{3}
    \]

(\mathbf{I} is the 3 × 3 identity matrix, \psi is the viscous term and \phi represents terms related to heat conduction, diffusion and work done by viscous forces)
The Magnetohydrodynamic Approximation

Electrodyamics governed by Maxwell’s Equations

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{4}
\]

\[
\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} + \mathbf{J} = 0 \tag{5}
\]

\[
\nabla \cdot \mathbf{D} = \rho_e \tag{6}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{7}
\]

supplemented by the equation for charge conservation:

\[
\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{8}
\]

and the generalized Ohm’s law which can be expressed as

\[
\mathbf{J} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{9}
\]
Simplifications and the Ideal MHD Equations

Simplifying Assumptions:
- Neglecting of Displacement Current - valid for low frequencies (till microwave range)
- Macroscopic Neutrality ⇒ neglect electrostatic body forces and convection current
- Dissipative effects (viscosity, conductivity, electrical resistivity) neglected

Ideal Magnetohydrodynamic Equations

\[ \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \]  \hspace{1cm} (10)

\[ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{uu} + P^* \mathbf{I} - \frac{\mathbf{BB}}{\mu}) = 0 \]  \hspace{1cm} (11)

\[ \mathbf{B}_t + \nabla \cdot (\mathbf{uB} - \mathbf{Bu}) = 0 \]  \hspace{1cm} (12)

\[ E_t + \nabla \cdot [(E + P^*) \mathbf{u} - \frac{1}{\mu} (\mathbf{u} \cdot \mathbf{B}) \mathbf{B}] = 0 \]  \hspace{1cm} (13)

\[ P^* = p + \mathbf{B} \cdot \mathbf{B} / 2\mu \] (full pressure = gas pressure + magnetic pressure),

\[ E = \rho \mathbf{u} \cdot \mathbf{u} / 2 + p / (\gamma - 1) + \mathbf{B} \cdot \mathbf{B} / 2\mu \] (total energy)

Additionally, \( \nabla \cdot \mathbf{B} = 0 \) (Initial Condition)
Effect of Magnetic Field

- Magnetic force is the gradient of the Magnetic stress tensor

\[
\mathcal{T}_m = \frac{1}{\mu} (\mathbf{B}\mathbf{B} - \frac{1}{2} B^2 \mathcal{I})
\]  

(14)

- In principal coordinates with \( \mathbf{B} = Bz \), it is equivalent to

\[
\mathcal{T}_m = \frac{1}{\mu} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & B^2
\end{bmatrix} + \frac{1}{2\mu} \begin{bmatrix}
-B^2 & 0 & 0 \\
0 & -B^2 & 0 \\
0 & 0 & -B^2
\end{bmatrix}
\]

(15)

- Effect of the magnetic force on fluid elements
  = isotropic magnetic pressure \((B^2/2\mu)\) + tension \((B^2/\mu)\) along field lines
- Freezing of magnetic field lines to the fluid ⇐ conservation of magnetic flux
  - Unrestricted movement of fluid along the field
  - Motion transverse to the field carries the field
MHD Waves

- Two types of wave motion possible: **longitudinal** (acoustic waves) and **transverse** (Alfvén waves)
- Alfvén Waves: equivalent to vibration of elastic cords under tension (due to tensile effect of the magnetic field)

\[ V_A = \left( \frac{\text{tension}}{\text{density}} \right)^{1/2} = \left( \frac{B^2}{\mu \rho} \right)^{1/2} \]  

(16)

- Carries perturbations in the transverse components of magnetic field and velocity
- Magneto-acoustic waves: similar to sound waves in gasdynamics
  - Parallel to magnetic field: same as gasdynamic sound waves (magnetic field does not affect flow)
  - Perpendicular to magnetic field: in addition to
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Figure: Parallel to Magnetic Field

Figure: Perpendicular to Magnetic Field
MHD Waves

- **Parallel to the Magnetic Field** - Two waves can exist
  - Alfvén Waves (transverse magnetic field and velocity components vary): wavespeed is
    \[ A = \sqrt{\frac{B^2}{\mu \rho}} \]
  - Acoustic Waves (pure sound waves): wavespeed is
    \[ a = \sqrt{\frac{\gamma p}{\rho}} \]

- **Perpendicular to the Magnetic Field** - Only the magneto-acoustic mode exists with wavespeed
  \[ V = \sqrt{a^2 + A^2} \]

- **Propagation along arbitrary direction** - Small perturbation equations give three solutions (slow MHD wave, fast MHD wave and Alfvén wave) with wavespeeds
  \[ c_a = A \cdot k = \frac{B \cos \theta}{\sqrt{\rho \mu}} \quad (A = \frac{B}{\sqrt{\rho \mu}}) \quad (18) \]
**Figure**: Alfven Speed greater than speed of sound

**Figure**: Alfven Speed less than speed of sound
1D MHD Equations

- The 1D system can be obtained by assuming that the gradients exist only along the x-direction.
- 1D MHD system in conservative form is:

\[ u_t + f(u)_x = 0 \]  \hspace{1cm} (20)

where \( u \) is the conserved vector and \( f(u) \) is the flux vector.

\[
\begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho w \\
B_y \\
B_z \\
E
\end{bmatrix}, \quad
\begin{bmatrix}
\rho u \\
\rho u^2 + P^* \\
\rho uv - B_y B_x \\
\rho uw - B_z B_x \\
uB_y - vB_x \\
uB_z - wB_x \\
(E + P^*)u - B_x(uB_x + vB_y)
\end{bmatrix}
\]  \hspace{1cm} (21)

- Zero divergence constraint \( \Rightarrow B_x = \text{constant} \)
- Flux \( f(u) \) is non-convex and non-strictly hyperbolic.
1D MHD Eigenstructure

- The 1D MHD system admits 7 eigenvalues
  - **Entropy wave** with wavespeed $\lambda_e = u$
  - two **Alfvén waves** with wavespeeds $\lambda^\pm_a = u \pm c_a$
  - two **fast magneto-sonic waves** with wavespeeds $\lambda^\pm_f = u \pm c_f$
  - two **slow magneto-sonic waves** with wavespeeds $\lambda^\pm_s = u \pm c_s$

where

$$c_a = \frac{B_x}{\sqrt{\rho}}$$  \hspace{1cm} (22)

$$c_{f,s}^2 = \frac{1}{2} \left[ \frac{\gamma \rho + B \cdot B}{\rho} \pm \sqrt{ \left( \frac{\gamma \rho + B \cdot B}{\rho} \right)^2 - \frac{4 \gamma \rho B_x^2}{\rho^2} } \right]$$  \hspace{1cm} (23)

- **Coincidence of Eigenvalues:**
  - $B_x = 0$ (Propagation perpendicular to the magnetic field): $c_a = c_s = 0 \rightarrow u$ is an eigenvalue with multiplicity 5
  - $B_y^2 + B_z^2 = 0$ (Propagation parallel to the magnetic field): $c_a, c_s$ are real and $u$ is an eigenvalue with multiplicity 3

Roe and Balsara's Eigensystem used for the present study
Numerical Scheme in 1D

- **Semi-discrete form** is
  \[
  \frac{du_i}{dt} + \frac{1}{\Delta x} \left( f_{i+1/2} - f_{i-1/2} \right) = 0 \tag{24}
  \]

- **Spatial Reconstruction**: done using an upwinded scheme

- **Time Evolution**: Runge-Kutta time-stepping usually used
The multi-dimensional MHD system can be written in the **conservative form**

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F} = 0 \quad \text{or} \quad \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} + \frac{\partial \mathbf{h}}{\partial z} = 0 \] (25)

**Non-strictly hyperbolic system** with complete set of eigenvectors

**Eight eigenvalues** corresponding to entropy wave, left and right moving slow, fast and Alfvén waves and zero

**2D Ideal MHD equations:**

\[
\begin{bmatrix}
\rho u \\
\rho u^2 + P^* - B_x^2 \\
\rho uv - B_y B_x \\
\rho uw - B_z B_x \\
0 \\
u B_y - v B_x
\end{bmatrix}, \quad
\begin{bmatrix}
\rho v \\
\rho v^2 + P^* - B_y^2 \\
\rho vw - B_z B_y \\
v B_x - u B_y \\
0
\end{bmatrix}
\]
Multi-dimensional algorithms: To ensure solenoidal nature of magnetic field (in multi-dimensions, $\sum_{faces} B_n \cdot ds = 0$)

- **Projection Scheme**: Poisson equation is solved to subtract the magnetic field with non-zero divergence

- **Constrained Transport / Central Difference (CT/CD)** using a staggered mesh

- **Eight-wave formulation**: Multi-dimensional MHD system modified to include source terms proportional to $\nabla \cdot B$ (non-conservative)

**Eight-Wave Formulation**: derived from governing equations by adding term proportional to $\nabla \cdot B$ as source term

$$\frac{\partial u}{\partial t} + \nabla \cdot F = S \text{ where } S = -(\nabla \cdot B)[0 \ B u \ u \cdot B]^T$$

(27)
The governing equation, discretized in space is given as:

\[
\frac{d\mathbf{u}_{ij}}{dt} V_{ij} + \sum_{\text{faces}} \mathbf{F} \cdot \mathbf{n} dS = \mathbf{S}_{ij} V_{ij} \Rightarrow \frac{d\mathbf{u}_{ij}}{dt} = \mathbf{Res}(i, j) \quad (28)
\]

where the residual is given by (for a quadrilateral cell)

\[
\mathbf{Res}(i, j) = -\frac{1}{V_{ij}} \left[ \sum_{l=1}^{4} \mathbf{F}_l \mathbf{n}_l dS_l + \mathbf{s}_{ij} \sum_{l=1}^{4} \mathbf{B}_l \mathbf{n}_l dS_l \right] \quad (29)
\]

where \( \mathbf{s} = [0 \ \mathbf{B} \ \mathbf{u} \ \mathbf{u} \cdot \mathbf{B}]^T \)

- **Spatial Reconstruction**: Upwinded reconstruction based on characteristic decoupling
- **Time Evolution**: Done using Runge-Kutta family of schemes
Some Problems and Applications

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Figure: Cloud Shock Interaction

Figure: Orszag Tang Vortex
Some Problems and Applications

Figure: Planetary Interaction with Solar Wind
Some Problems and Applications

Figure: Blunt Body Computations - Shock Stand-Off Distances
Conclusion

- Lots of unresolved issues still remain in MHD
- **Lack of Roe-type averaging** for flux computation
  - Roe-type averaging possible only for $\gamma = 2$
  - Arithmetic averaging used by all Roe-type schemes
- **Admissibility of “exotic shocks”:**
  - **Non-Convex Flux function:** Rankine-Hugoniot conditions admit “**intermediate waves**” across which only one family of characteristics converge
  - Admissibility is debated $\rightarrow$ straightforward extension of admissibility conditions of convex systems does not work
- MHD system **highly non-linear** $\rightarrow$ Higher order schemes yield oscillatory solutions, even at low CFL $\rightarrow$ most schemes used till now are highly TVD