AE 415/713 — Spaceflight Mechanics, Spring 2008

Question Bank

Possibly useful data: $G = 6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2$, mass of the earth $M_\text{e} = 5.98 \times 10^{24}\text{kg}$, one sidereal day = 23 hours, 56 minutes and 4.09 seconds, radius of the earth = 6378 km, mean solar day = 24 hours, $\mu_\text{sun} = 1.32715 \times 10^{11}\text{km}^3/\text{s}^2$, $\mu_\text{earth} = 3.98601 \times 10^5\text{km}^3/\text{s}^2$, 1 AU = 1.49599×10^8 km.

Orbital Mechanics

1. The perihelion and aphelion distances of the earth are 147.5 and 152.6 million kilometers, repectively. Find the eccentricity of the earth's orbit and the maximum speed of earth in its orbit. (Hint: One year equals 365.25 *mean solar* days.)

2. A projectile is launched 500 km above the earth's surface at a right ascension of 45° and latitude of 30°. The initial velocity satisfies $-\dot{x} = \dot{y} = \dot{z}$. The initial speed is 7.6 km/s. Find Ω, i, ω, e and a for the resulting orbit. If the orbit is periodic, find the period. Take the radius of the earth to be 6378 km.

3. A satellite is released in the equatorial plane at a height of 1000 km above the earth's surface and parallel to it. The initial velocity vector makes an angle of 45° into the northern hemisphere w.r.t. the equatorial plane. At what speed must it be released so that it moves in an elliptical orbit of eccentricity 0.2? If the right ascension of the point of release is 30°, estimate the geocentric position components of the satellite two hours after release. (Hint: One iterative procedure for solving Kepler's equation for elliptical orbits is to form the sequence of approximations $E_{i+1} = M + e \sin E_i$ starting from some initial guess E_0. It can be shown that this sequence of approximations converges to the unique solution E of Kepler's equation at a rate such that $|E_i - E| \leq e^i(1-e)^{-1}|E_1 - E_0|$.)

4. A satellite is observed to pass over the equator at an altitude of 1,000 km and right ascension of 30° with a speed of 9 km/s. The velocity vector makes an angle of 10° above the geocentric xy plane, while the projection of the velocity vector in the geocentric xy plane makes an angle of 130° with the vernal equinox. Find the geocentric position components of the satellite 3 hours after this sighting.

5. A satellite is observed to pass over the geographic north pole at an altitude of 1,000 km and with a speed of 9 km/s. The velocity vector makes an angle of 10° above the geocentric xy plane, while the projection of the velocity vector in the geocentric xy plane makes an angle of 30° with the vernal equinox. Find the geocentric position components of the satellite 3 hours after this sighting.

6. A earth satellite crosses the equator going south to north at a point having longitude 22° west. Three hours, 58 minutes and 20.68 seconds later, the satellite passes overhead a point having longitude 102° east and latitude 25° north. Find Ω and i for the orbit. Assume that the Greenwich meridian has a right ascension of 45° at the time of the first observation.

7. The satellite in the question above next crosses the equator going south to north at a point having longitude 52° west. Find the semimajor axis of the satellite orbit.

8. The maximum and minimum rates of rotation of the line joining the center of the earth (origin of the geocentric coordinate system) to the satellite in the two questions above are estimated to be 0.0575 and 0.0442 deg/s, respectively. Estimate the eccentricity of the orbit.

9. Find the latitude λ of a satellite in terms of its true anomaly ν and the six classical elements Ω, i, ω, a, e and t_p. Use the expression you get to show that $\lambda \leq i, \ \pi - i$.

10. Halley's comet last passed perihelion on February 9, 1986. It has a semimajor axis of 17.9564 AU, and an eccentricity of 0.967298. Estimate its distance from the sun today.

11. A spacecraft moving solely under the influence of the sun has initial position components $x(0) = 1.2$ AU, $y(0) = z(0) = 0$, and initial velocity components $\dot{x}(0) = \dot{y}(0) = 25$ km/s, and $\dot{z}(0) = 18.7$ km/s. Is the resultng orbit closed? If so, find the elements of the orbit. Measure inclination from the $x - y$ plane, and measure right ascension from the x axis.

12. A projectile is injected into the earth's gravitational field at a geocentric distance r with speed v, such that the angle between the radius vector and velocity vector is $\beta + (\pi/2)$. Show that the resulting orbit has semimajor axis and eccentricity given by

$$a = \frac{\mu r}{2\mu - rv^2}, \quad e^2 = \sin^2\beta + \cos^2\beta\left(1 - \frac{r^2v^2}{\mu}\right)^2.$$

Find an expression for the initial true anomaly.

13. A projectile is fired from the surface of a homogeneous spherical planet of radius r with speed v and elevation $\beta > 0$. For what combinations of v and β will the projectile impact the planet? Give formulae that can be used to compute the range (distance along the planet's surface) to the point of impact, the time of flight, and the maximum altitude attained by the projectile. (The notation refers to the previous question.)

14. The polar equation of an orbit relates the radial distance to the true anomaly. Derive an expression relating the radial distance along an elliptical orbit to the eccentric anomaly.

15. Find the shortest possible time taken by a satellite in an elliptical orbit of semimajor axis a and eccentricity e to move from one end of the minor axis to the other.

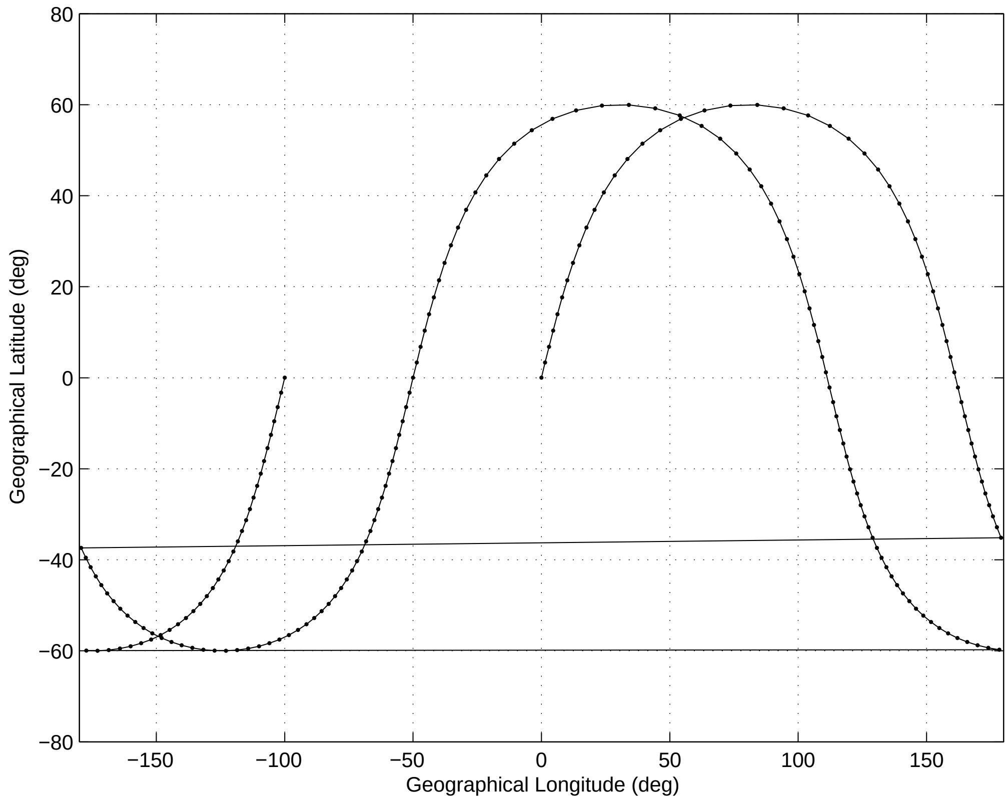
16. An asteroid is in an elliptical of semimajor axis a_1 and eccentricitry e_1 around the sun. An impulsive impact causes the speed of the asteroid to instantaneously change from v to αv with no change in the velocity direction when the asteroid is located along the minor axis of the orbit. Find bounds on the factor α that guarantee a bounded orbit. Find the semimajor axis a_2 and the eccentricity e_2 of the new orbit. Show that the impact causes the major axis of the orbit to rotate by an angle $\cos^{-1}(e_1/e_2)$.

17. A missile warning system detects an unidentified object at an altitude of $0.5R_\mathrm{e}$ moving with an instantaneous speed of $(\sqrt{2/3})v_\mathrm{e}$ at a flight path angle of 30° (above the local horizontal plane), where R_e is the radius of the earth, and v_e is the speed along a circular orbit just grazing the earth's surface. Is the object likely to be a deep-space probe, an earth satellite, or a ballistic missile?

18. An elliptical orbit of semimajor axis a and eccentricity e intersects a circular orbit at a point A such that the speeds at point A along both the orbits are the same. Both orbits are about the same primary mass. Find the time taken to travel along the elliptical orbit between the periapsis and the point A. What is the radius of the circular orbit?

19. The figure below shows the *ground trace* (path traced on the earth by the point directly under the satellite) of a satellite. The dots indicate positions separated by equal intervals of time. Estimate from the ground trace the semimajor axis, the inclination and the argument of perigee of the satellite orbit. If the right ascension of the Greenwich meridian is 60° at $t = 0$, then also estimate the right ascension of the ascending node.

How many orbits will it take before the ground trace starts repeating itself? Neglect perturbation effects.



20. A deep space probe is at a distance of one atronomical unit (AU) from the sun in a circular orbit. The mission requires the probe to leave the solar system along a direction that makes an angle of 45° with its velocity vector at a certain point A in the circular orbit. Find the increase in speed required at A if the direction of the initial velocity is to remain the same.

Satellite Operations

21. A *geosynchronous* satellite is a satellite whose orbital period is equal to one *sidereal* day (period of rotation of earth – 23 hours, 50 minutes, 4.09 seconds). What is the semimajor axis of the orbit of a geosynchronous satellite? A *geostationary* satellite is a satellite that appears stationary relative to earth. What are i and e for such a satellite? What is its speed at perigee? Take $G = 6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2$ and mass of earth $M_{\text{e}} = 5.98 \times 10^{24}\text{kg}$.

22. Show that the smallest possible inclination angle of the orbit of a satellite is equal to the latitude of the satellite at the point of release. (For this reason, it is more economical to launch communications satellites from as close to the equatorial plane as possible). Show that if the perigee of such an orbit lies on the equatorial plane, then the true anomaly of the satellite at injection is 90°.

23. An earth satellite is to be injected into a geostationary transfer orbit (GTO) from an altitude of 200 km and a latitude of 23° N such that the orbit has the smallest inclination possible, and the apogee of the orbit lies in the equatorial plane and has geostationary altitude. Assuming that the injection point has a right ascension of 120° with respect to the vernal equinox, find the geocentric components of the required initial velocity vector.

24. The satellite in Problem 19 is to be transferred from the GTO to a geostationary orbit using a single impulsive burn. When and where do you recommend that this transfer

should be performed? Calculate the change in the velocity vector (magnitude and direction w.r.t. the equatorial plane) that should be imparted to the satellite to transfer it into a geostationary orbit. Calculate the shortest time following GTO injection after which the transfer may be performed.

25. For the same initial data as in Problem 19, find the classical elements for a GTO that has the smallest possible inclination subject to the constraint that every point on the orbit has an altitude of at least 100km.

26. Satellites used for studying earth's resources are usually put in polar orbits. Why?

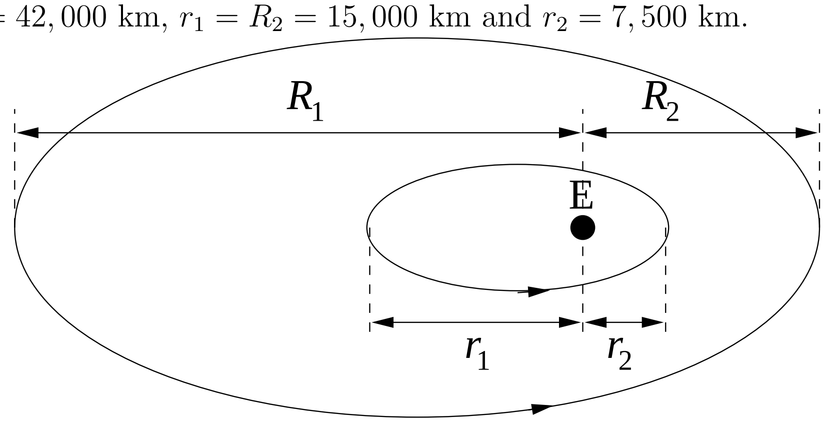
27. Find the area on the surface of the earth that is visible from a satellite placed at an altitude h above the surface. Find the area on the surface of the earth from which a satellite placed at an altitude h is visible at an elevation angle not less than β.

Orbital Maneuvers

28. A supply ship and a space station are in the same circular orbit of radius a about the earth. Due to (huge) errors at the time of injecting the supply ship, it follows the space station 30 degrees behind along the orbit. Devise an orbital maneuver involving one impulsive thruster firing so that the supply ship intercepts the space station. Neglect perturbation effects.

29. A satellite moves in an orbit of semimajor axis a and eccentricity e. When the satellite is at the semilatus rectum, an impulsive maneuver is performed to rotate the semimajor axis of the orbit by 90° without changing its eccentricity. Calculate the components of the impulse required in the perifocal frame of the original orbit, and the semimajor axis of the new orbit.

30. The figure below depicts two coplanar coaxial elliptical orbits around the earth (marked E). Design a sequence of maneuvers for transferring from the smaller orbit to the larger one using impulsive thruster firings. Give the velocity increments/decrements to be imparted, locations where thrusters should be fired, and time durations between firings. Take $R_1 = 42,000$ km, $r_1 = R_2 = 15,000$ km and $r_2 = 7,500$ km.



31. An asteroid is in an elliptical of semimajor axis a and eccentricitry e around the sun. An impulsive impact with a second asteroid causes the first asteroid to change to a circular orbit of radius a. Show that the impact must have occurred when the first asteroid was on the minor axis of its orbit. Show also, that the magnitude of the velocity change caused by the impact must be $\sqrt{(\mu/a)}(\sqrt{1+e} - \sqrt{1-e})$.

32. It is desired to tilt the orbital plane of a satellite by an angle ϕ without changing the perigee and the apogee positions. Find the required velocity increment (magnitude and direction) in terms of the speed at perigee if the maneuver is performed at the perigee. At which location will the maneuver consume more energy, apogee or the perigee?

33. Two asteroids A and B move in coplanar circular orbits of radii 2 AU and 3.5 AU, respectively, about the sun. At $t = 0$, the position vectors of the asteroids make angles 139° and 271°, respectively, w.r.t. a reference line in the orbital plane. A spaceship has to take off from A and travel to B using the least amount of fuel possible. Find the first available take-off time in days from $t = 0$. Assume the gravitational fields of the asteroids to be negligible. (Hint: The Hohmann transfer is fuel optimal among impulsive transfers between coplanar circular orbits.)

Perturbed Orbital Motion

34. (**An application of regression of nodes.**) A *sun-synchronous* orbit is an orbit whose rate of nodal regression is equal to 360° per year. Satellites used for photographing the earth's surface are usually put in a sun-synchronous orbit. Why?

35. It is desired that the *ground trace* of an earth observation satellite should repeat after every N orbits, so that the satellite takes the same sequence of photographs. It is also desired that every photograph of a given location on the ground trace is taken under the same lighting conditions. Find a and i for the orbit, if the orbit is to be circular.

36. The Landsat 7 satellite is a earth observation satellite in a sun-synchronous orbit. The satellite repeats the same set of observations every 16 days or 233 orbits. Find the orbital period and a and i for the orbit. Check your answer at http://landsat7.usgs.gov/index.html. The website says that the descending node of the orbit is at 10-10:15 am local time. What does it mean?

37. The IRS-P4, also called OCEANSAT, is a 1050 kg Indian remote sensing satellite used for ocean applications. OCEANSAT operates in a circular sun-synchronous orbit at an altitude of 720 km. Find the orbital inclination and period of the satellite. OCEANSAT was launched into orbit by the Indian Polar Satellite Launch Vehicla (PSLV). Find the speed of the PSLV at the time of orbit injection.

38. An earth observation satellite is placed in a circular sun-synchronous orbit at an altitude such that the ground trace repeats itself every day after crossing the equator sixteen times. Find the inclination of the orbit and the altitude of the satellite. What minimum field of view should the satellite have so that every point on the equator comes within the field of view at least once every day?

39. The Indian remote sensing satellite Cartosat 1 is in a circular sun synchronous orbit having an orbital period of 97 minutes. Find the altitude and inclination of the orbit. After what time does the ground trace of the satellite start repeating itself?

40. (**An application of advance of perigee**) Large parts of Russia are inaccessible to geostationary communication satellites because of high latitudes. In order to cater to such areas, a series of Russian communication satellites used relatively highly eccentric orbits (why large eccentricity?) with the apogee placed over the region that the satellites serve (why?). However, if the apogee drifts due to earth's oblateness, then the satellites will be unavailable for communication for large periods of time. How do you think this problem can be solved? For more details on this series of satellites, see http://www.astronautix.com/project/molniya.htm.

Launch to Rendezvous

41. A supply ship is to be launched from a launch site located at 13.75° N and 80.25° E to dock with a space station in a circular orbit of altitude 1,000km, inclination 25° and right ascension of ascending node 55°. If the ascending node of the orbit is at 0° longitude

at $t = 0$, find the first two available launch times. Find the launch azimuths required if the supply ship is to be launched at those times.

42. The space shuttle launch site at Cape Canaveral, Florida, has latitude $28^\circ\ 30'$ N and longitude $80^\circ\ 33'$ W. Safety considerations restrict the launch azimuth (measuerd from the local north) to lie between 35° and 120°. Find the maximum and minimum orbital inclinations that can be achieved by launching from Cape Canaveral, without considering any plane change maneuvers. ($1^\circ = 60'$.)

43. The Russian Progress spacecraft is often used to deliver supplies to the International Space Station (ISS), which is placed in a circular orbit of altitude 344 km and inclination 51.51°. The Progress spacecraft is launched from the Baikonur Cosmodrome (45.92° N, 63.34° E) in Kazakhstan using a Soyuz rocket. If the ascending node of the ISS orbit is at 60° W longitude and 333.61° right ascension at $t = 0$, find the first two available launch times for the Progress spacecraft. Find the launch azimuths required if the Progress spacecraft is to be launched at those times.

Multi-stage Rockets and Parallel Staging

44. GSLV-D1, the first developmental flight of the Indian GSLV, consisted of three stages. The first stage GS1 consisted of a S125 solid stage having a total mass of 156 tonnes, and four L40 boosters strapped on to the S125. The S125 has a solid propellant motor that burns 129 tonnes of HTBP propellant in 100 seconds generating 4700 kN of thrust. Each L40 has a total mass of 46 tonnes, and a liquid propellant engine that burns 40 tonnes of UDMH/N_2O_4 propellant in 160 seconds generating 680 kN of thrust. The second stage GS2, which ignites immediately after S125 burns out, has a total mass of 42.8 tonnes, and uses a liquid propellant engine that consumes 38 tonnes of UDMH/N_2O_4 propellant in 150 seconds producing 720 kN of thrust. The third stage GS3 has a total mass of 15 tonnes and uses a cryogenic engine that consumes 12.5 tonnes of liquid hydrogen and oxygen in 720 seconds producing a thrust of 73.5 kN. GSLV-D1 also carried an experimental communications satellite GSAT-1 weighing 1530 kg. Find the maximum possible velocity that GSAT-1 could have achieved. (Note: S125 and the L40's are ignited and ejected together. Total mass includes propellant.)

45. India's Polar Satellite Launch Vehicle (PSLV) injected the Indian remote sensing satellite IRS-P4, the Korean KITSAT-3 and the German DLR-TUBSAT into a polar sun synchronous orbit on its second operational launch in May 1999. The PSLV has four stages that use solid and liquid propellant alternately. The first stage engine burns 138 tonnes of solid Hydroxyl Terminated Poly Butadiene (HTPB) in 107.4 seconds to produce a maximum thrust of 4628 kN. The first stage has six strap-on boosters which are ignited at launch. Each booster consumes 9 tonnes of HTPB propellant in 45 seconds to produce a maximum thrust of 662 kN, and is ejected **along with** the first stage after the first stage burns out. The total mass of the first stage (including structure, propellant and boosters) is 229 tonnes. The second stage engine burns 40.6 tonnes of liquid Unsymmetrical Di-Methyl Hydrazine (UDMH) propellant and nitrogen tetroxide (N_2O_4) oxidizer in 163 seconds to generate a maximum thrust of 725 kN. The third stage engine burns 7.2 tonnes of solid HTPB propellant in 76 seconds to yield a maximum thrust of 340 kN, while the twin engines of the fourth stage together consume 2 tonnes of liquid Mono-Methyl Hydrazine (MMH) propellant and oxidizer in 415 seconds to produce a total maximum thrust of 14.8 kN. The total masses (structure + propellant) of the second, third and fourth stages are 46 tonnes, 8.4 tonnes and 2.89 tonnes, respectively. The masses of the IRS-P4, KITSAT-3 and DLR-TUBSAT are 1050kg, 107kg and 45

kg, respectively. Find the maximum possible burnout speed of the PSLV under ideal conditions.

46. On January 25, 1994, the Titan 23G rocket carried NASA's 227 kg spacecraft Clementine to space. Titan 23G is a two stage liquid propellant rocket. The first stage has a dry (empty) mass of 4,760 kg, and its twin LR-87 engines generate a total thrust of 2100 kN for 156 seconds at a specific impulse of 258 seconds each. The second stage has a dry mass of 2,760 kg, and its single LR-91 engine generates a thrust of 440kN for 180 seconds at a specific impulse of 316 seconds. Find the ideal burnout speed of the Titan 23G rocket at the time of the Clementine launch.

47. Each of the two solid boosters of the space shuttle burn 5,02,700 kg of propellant in 2 minutes starting from launch. Each booster has a dry mass of 81,900 kg and a relative exit velocity of 2812.6 m/s. Starting from launch, the main engines on the orbiter, having a relative exit velocity of 4459 m/s, burn liquid propellant from the external fuel tank at the rate of 1464 kg/s. The external fuel tank has a dry weight of 35,000 kg and contains 7,03,000 kg of liquid propellant. When the external tank runs out, it is discarded, and the orbiter continues till the orbit. The orbiter weighs 75,200kg (nett) and carries a payload of 29,500 kg in addition to 3,600 kg of liquid propellant. Find the burnout velocity of the orbiter in vacuum in the absence of gravity. ("Nett" and "dry" refer to structural mass alone.)

Optimal Staging and Trade-off Ratios

48. The paint applied to the external fuel tank of the space shuttle described in the problem above is found to weigh about 1000 kg. How much payload benefit/penalty will result if the external fuel tank is not painted?

49. The data given below pertains to a three-stage rocket carrying a payload of 35,400 kg. Take $g = 9.8\text{m/s}^2$.

	Stage 1	Stage 2	Stage 3
Structural mass (kg)	6,98,390	69,840	21,950
Propellant mass (kg)	47,61,750	6,98,400	69,480
$-\dot{m}$ (kg/s)	55,300	5,533	700
Specific impulse (s)	230	286	286

(a) If each stage ignites after the previous one has burnt out, calculate the burnout velocity of the rocket in free space.

(b) If the specific impulse of each of the two upper stages is increased to 295 s at an expense of a seven percent increase in the structural mass of each of the two upper stages, how would the payload change for the same burnout velocity?

(c) This question relates only to the last two stages. Redesign the third stage (that is, find structural and propellant masses) so that the overall payload ratio for the last two stages is maximised for the same velocity increment (as in question a) over the last two stages. Assume the same structural ratio for the redesigned third stage as the second stage. What is the new payload mass?

50. Each of the two stages of a two-stage rocket has a specific impulse of 286 seconds, and a structural ratio of 0.1. The rocket is optimally sized to deliver maximum payload at a burnout velocity of $7,000$ m/s in free space. If the first stage has a structural mass of 100 tonnes, find the payload mass, the structural mass of the second stage, and propellant masses of both stages. Estimate the change in payload that will result for the same burnout velocity if the structural mass of the first stage is increased by 500 kg.

51. Find the sensitivity of the payload mass to a change in the engine performance (reflected as a change in the relative exit velocity) of the kth stage of a multistage rocket assuming the final burnout velocity to remain the same. A two-stage rocket carries 1,167 kg of propellant in its first stage and 415 kg in its second. The structural masses of the two stages are 113 kg and 41 kg, respectively. The payload weighs 150 kg. Both stages have a specific impulse of 282 sec. Engine improvement can lead to 10 s increase in the specific impulse. If the improvement can be made only to one stage, in which stage would it be most beneficial to introduce the improvement?

52. All stages of an N stage rocket are to have the same structural ratio ϵ and the same relative exit velocity V_e. The stagewise payload ratios $\Pi_1, \ldots \Pi_N$ are to be designed such that the burnout velocity (in free space) is maximized subject to the constraint that the overall payload ratio equals a given value Π_*. Show that all stagewise payload ratios have to be equal. Find the common value of the optimal stagewise payload ratios and the optimum value of the burnout velocity in terms of the given data.

Rectilinear Rocket Trajectories

53. Two identical rockets ascend vertically in a uniform gravitational field. The first ascends at constant thrust, while the second ascends at a constant specific thrust equal to the initial specificthrust of the first. Which rocket achieves a higher burnout altitude?

54. A two-stage sounding rocket ascends vertically in uniform gravity at a constant specific thrust of ψ for both stages. Two kinds of ascent trajectories are being considered. In the first trajectory, the second stage ignites immediately after the first stage burns out and has been discarded. In the second trajectory, the second stage is not ignited until it achieves its maximum altitude after the first stage burns out and has been discarded. For which trajectory does the payload achieve a higher culmination altitude?

55. Two identical sounding rockets ascend vertically from rest under uniform gravity in vacuum. One follows a constant specific thrust profile, while the other follows a constant thrust profile. If both rockets have the same average thrust over their respective burn times, then show that both rockets will achieve the same burnout velocity. Which rocket will achieve a higher burnout altitude?

56. A sounding rocket ascends vertically under uniform gravity at constant specific thrust ψ. Find the values of ψ required to maximize a) the burnout altitude and b) the culmination altitude (keeping everything else the same). Find the corresponding values of the respective altitudes and burn times. Your answers will indicate why fireworks are launched using an *impulsive shot*, in which all the propellant is consumed instantaneoulsy upon launch.

Gravity Turn Trajectories

57. A single stage rocket follows a gravity-turn trajectory at zero angle of attack (thrust aligned with velocity) and constant specific thrust ψ in a uniform gravitational field in vacuum. If the speed and flight path angle at the start t_0 of the turn trajectory are V_0 and γ_0, respectively, then show that

$$V(t)\cos\gamma(t)\left[\frac{1+\sin\gamma(t)}{1-\sin\gamma(t)}\right]^{\frac{\psi}{2}} = V_0\cos\gamma_0\left[\frac{1+\sin\gamma_0}{1-\sin\gamma_0}\right]^{\frac{\psi}{2}},$$

$$g(\psi^2-1)(t-t_0) = V(t)(\psi+\sin\gamma(t)) - V_0(\psi+\sin\gamma_0),$$

where g is the gravitational acceleration. (Hint: Differentiate.)

58. A single stage rocket takes off vertically under uniform gravity. At a certain altitude, a gravity-turn trajectory is initiated by turning the velocity vector through an angle of 5° from the vertical. The turning phase takes place at zero angle of attack and a constant specific thrust of 2.5. The burnout velocity and the burnout flight path angle are required to be 3650 m/s and 55°, respectively. The burnout mass of the rocket is 3000 kg.

    (a) What should be the velocity of the rocket at the start of the turning trajectory?

    (b) At what altitude should gravity turning be initiated so that the burnout altitude is 373 km?

    (c) Calculate the burnout downrange.

    (d) The vertical ascent is also to take place at constant specific thrust. Find the value of specific thrust required for the vertical ascent phase.

    (e) Find the total flight time, and the mass of propellant required.

    Assume that $I_{sp} = 300$s for the entire flight. You may neglect atmospheric drag and the curvature of the earth if you wish.

59. A single stage rocket takes off vertically under uniform gravity. At a certain altitude, a gravity-turn trajectory is initiated by turning the velocity vector through an angle of 4° from the vertical. The turning phase takes place at zero angle of attack and a constant specific thrust of 2.5. The burnout velocity and the burnout flight path angle are required to be 4000 m/s and 45°, respectively. The burnout mass of the rocket is 3000 kg. What should be the velocity of the rocket at the start of the turning trajectory? Find the flight time and the propellant required for the turning phase of the trajectory. Assume that $I_{sp} = 300$s for the entire flight. You may neglect atmospheric drag and the curvature of the earth if you wish.

60. A single-stage rocket takes off vertically. At an altitude of 1 km, a gravity turn is initiated by turning the velocity vector through an angle of 5°. The desired flight path angle at burnout is 55°, while the final mass of the rocket at burnout is 3000 kg. Assuming that the entire flight occurs at a constant specific thrust of 2.5, a specific impulse of 300 s, and zero angle of attack, find the mass of propellant required, speed, altitude and down range at burnout, and the total burn time. Neglect drag and curvature of the earth.

61. A rocket ascends in a uniform gravitational field (g=9.8m/s^2) in vacuum at a constant specific thrust of 2.5 and a constant specific impulse of 300s. The rocket weighs 20,000 kg at launch and carries 18,000kg of propellant. Design a pitch program that will attain a burnout flight path angle of 55° with maximum possible burnout speed. Calculate the burnout speed, burnout altitude and downrange for the pitch program that you design.

62. The rocket in the previous problem is to be prepared for a different mission. The new mission involves ascending vertically to an altitude of 1km, where a gravity turn is initiated by turning the velocity vector by an angle of 5°. What mass of propellant should the rocket carry so that it still achieves the same burnout flight path angle? Calculate the burnout speed, altitude and downrange, and the total burn time for the new mission.

Attitude Representation

63. A frame B is rotated through an angle of 45° about a unit vector having components $[1\ 0\ 0]^T$ in the frame B. The resulting frame B_1 is rotated through an angle of 30° about a vector having components $3^{-1/2}[1\ 1\ 1]^T$ in the frame B_1, to obtain a frame B_2. Find

a single rotation (axis and angle) that takes B to B_2. Compute the rotation matrices relating frames B to B_1, B_1 to B_2 and B to B_2. (Hint: Use problem 71.)

64. The unit quaternion that describes the orientation between two frames I and B is given by $0.5(1 + \mathrm{i} + \mathrm{j} + \mathrm{k})$. Find the rotation matrix describing the same orientation. (Hint: Use problem 71.)

65. A frame I is rotated through 90° about a unit vector having components $[1/\sqrt{2}, 1/\sqrt{2}, 0]^{\mathrm{T}}$ in the frame I to obtain a frame B_1. Frame B_1 is rotated about its z-axis through an angle of 60° to obtain the frame B. Find a single rotation (axis and angle) that takes the frame I to the frame B. Find the components in I of the unit vector aling the x-axis of frame B.

66. An orthogonal right-handed frame I formed by the unit vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ is rotated about the vector $\mathbf{l}+\mathbf{m}+\mathbf{n}$ through an angle 60° to obtain a frame B. Find the rotation matrix that transforms components of any vector in B to components of that same vector in I. Express the unit vectors of frame B in terms of $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$. (Hint: Use problem 71.)

67. Use quaternions to show that three successive rotations of 180° each about three mutually perpendicular axes returns a body to its original orientation.

68. An aircraft has, at a certain instant, a yaw angle $\psi = 120°$, pitch angle $\theta = 60°$ and roll angle $\phi = 90°$ with respect to a reference frame. Describe a single rotation (axis and angle) that makes the reference frame coincide with the aircraft frame. (Recall that the aircraft Euler angles are 3-2-1 Euler angles: the reference frame coincides with the aircraft frame when rotated through ψ about z, θ about y, and ϕ about x in that order.)

69. Unit vectors $\mathbf{i}$ and $\mathbf{j}$ of a frame B have components $[0\ 1/\sqrt{2}\ 1/\sqrt{2}]^{\mathrm{T}}$ and $[1/\sqrt{3}\ -1/\sqrt{3}\ 1/\sqrt{3}]^{\mathrm{T}}$, respectively, in frame I. Find the rotation matrix that can be used to transform the components of a vector in B to its components in I.

Attitude Kinematics

70. At the instant described in the problem 69 above, frame B is rotating relative to I such that the components of $\dot{\mathbf{i}}$ and $\dot{\mathbf{j}}$ in the B frame are $[0\ 1\ -2.4]^{\mathrm{T}}$ and $[-1\ 0\ 3.1]^{\mathrm{T}}$, respectively. Find the components of the angular velocity of B along unit vectors of B.

71. If $v = [v_1 v_2 v_3]^{\mathrm{T}}$ gives the components of a unit vector, show that $(v\times)^3 = -(v\times)$, where

$$(v\times) = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

A frame B rotates at a constant speed Ω about a fixed unit vector whose body components are given by v. Show that the rotation matrix describing the instantaneous orientation of B relative to its initial orientation is given by $R(t) = I + (1 - \cos\Omega t)(v\times)^2 + (\sin\Omega t)(v\times)$. (**Hint:** Substitute in the attitude kinematics equation.)

72. Frame B rotates relative to frame I such that the vector ω of B-frame components of the angular velocity of B relative to I satisfies $\dot{\omega} = a \times \omega$, where a is a constant vector fixed in the B frame. Describe the motion of ω (as seen in the B-frame). Show that the I-frame components of the vector $a + \omega$ remain constant.

73. Use the equations for rotational kinematics and dynamics of a torque-free rigid body to show that if the initial angular velocity is scaled by a factor α, then the attitude time history will be speeded up by the same factor α, but otherwise remain the same.

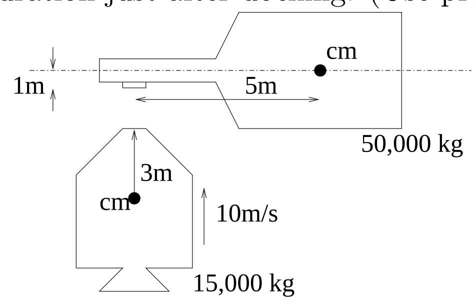
Attitude Dynamics

74. Two body fixed frames B and B′ are parallel. The origin of B′ has a coordinate vector $\rho \in \mathbb{R}^3$ in the frame B. Find the moment of inertia matrix $I_{B'}$ with respect to B′ in terms of I_B.

75. The figure below depicts a docking between a spacecraft and a space station. The moment of inertia matrices (in kg m^3) of the spacecraft and the space station about their respective principal axes are

$$I_{sc} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 20 \end{bmatrix} \times 10^3, \; I_{ss} = \begin{bmatrix} 90 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} \times 10^3.$$

Find the velocity of the center of mass and the angular momentum about the center of mass of the docked configuration. Find the moment of inertia matrix and the angular velocity of the docked configuration just after docking. (Use problem 74.)



76. It is well known that, in order to pull a coordinated turn at speed V and radius R, an aircraft has to bank at an angle $\phi = \tan^{-1} V^2/Rg$. Show that, to maintain the turn, the net external moment about the roll axis must equal $\frac{1}{2}(C - A)(V^2/R^2)\sin 2\theta$, where C and A are moments of inertia about the yaw and pitch axes, respectively. Assume the yaw and pitch axes to be principal axes.

77. A rigid satellite with negligible internal dissipation has principal moments of inertia 500, 600 and 300 kgm^2. The satellite initially rotates about the minor principal axis at a speed of 0.2 rad/s. Two masses 1kg each (not included in the inertias above) pull out two 50 metre wire antennas of negligible mass along the intermediate principal axis. The satellite is intended to spin about the original axis with the antennas taut and fully deployed. What is the design spin rate of the satellite? Will this scheme work as designed or is there a flaw in the scheme?

78. A body of mass M and principal moments of inetia A, B and C drops at a speed V and simultaneously spins at 10 rpm about an axis that makes equal angles with the three principal axes. Find the total kinetic energy of the body.

79. The x-axis of the principal body frame of a communications satellite in geostationary orbit always points along the orbital velocity, while communications equipment placed along the z-axis points towards the earth. What are the body components of the angular velocity? In your opinion, which axis should be designed to be the major principal axis and why? An attitude control gas-jet thruster located on the y-axis malfunctions causing a certain mass of gas to be impulsively ejected along the z axis. If the major principal moment of inertia is twice the value of each of the other two, then show that the body components of the angular velocity repeat after every orbit.

80. A rigid body has principal moments of inertia 2 kgm^2, 3 kgm^2 and 5 kgm^2. The body is spun about the unit vector $(\mathbf{i}+2\mathbf{j}+\mathbf{k})/\sqrt{6}$ at 5 rpm before being released, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the minor, intermediate and major principal axes of inertia, respectively. Find the rotational kinetic energy and magnitude of angular momentum in the body. Will the angular velocity vector trace out polhodes about the major axis or the minor axis for the given initial condition? Neglect dissipation.

Torque-free motion of an axisymmetric body

81. An axisymmetric body has principal moments of inertia $I_{xx} = I_{yy} = A$ and $I_{zz} = C$. The initial angular velocity vector has components $\omega(0) = [\omega_{10}, \omega_{20}, \Omega]^{\mathrm{T}}$ in the body-fixed principal axis frame. Show that, when viewed from the body-fixed principal axis frame, the angular velocity vector performs a coning motion around the axis of symmetry. Show that the cone traced out by the angular velocity vector has height Ω and a base of radius $\sqrt{\omega_{10}^2 + \omega_{20}^2}$. Show that the rate of coning is $\Omega(C-A)/A$. Show that the angular velocity vector always lies in the plane determined by the axis of symmetry and the angular momentum vector. (Hint: Guess $\omega(t)$ and verify by substituting into Euler's equation. Alternatively, consider Euler's equations in the prncipal frame, and calculate $\ddot{\omega}_1$ and $\ddot{\omega}_2$.)

82. Show that the axis of symmetry of the body in the problem above precesses (relative to an inertial observer) at a rate given by H/A, where H is the magnitude of the angular momentum vector. (Hint: Use the results of the previous problem along with the fact that the kinetic energy ellipsoid rolls without slipping on the invariant plane.)

83. A disc of negligible thickness is spun about an axis that makes an angle γ with the normal and then released. Show that the half angle θ of the cone generated by the disk normal is given by $\tan\theta = \frac{1}{2}\tan\gamma$.

84. A cylindrical disc of radius R and height $R/2$ rotates in free space such that its axis precesses generating a cone of half angle 15° as seen by an inertial observer. What is the half angle of the cone generated by the angular velocity as seen by an inertial observer?

85. A disc of negligible thickness when spun and thrown up is seen to wobble such that its normal traces a cone of 20° once every second. Determine the spin rate (component of angular velocity normal to the disc) and the magnitude of the angular velocity vector. (Use problem 82.)

86. The space cone and body cone of an axisymmetric body have a semi-vertex angle of 30° each. Find the ratio of the maximum principal moment of inertia to the minimum principal moment of inertia. Is the body prolate or oblate?

87. A slender axisymmetric missile has $I_{\mathrm{t}}/I_{\mathrm{s}} = 20$, where I_{s} is the moment of inertia about the axis of symmetry, while I_{t} is the moment of inertia about a transverse axis. The missile has a spin rate (component of angular velocity along the axis of symmetry) of 10π rad/s, and the axis of symmetry precesses making an angle of 5° with a fixed direction in space. Find the angle between the angular velocity vector and the axis of symmetry and the magnitude of the angular velocity vector. Draw the space and body cones for the missile.

88. A communications satellite in geosynchronous orbit has principal moments of inertia $I_{xx} = I_{zz} = 50{,}000\text{kgm}^2$ and $I_{yy} = 100{,}000\text{kgm}^2$. The x axis of the principal axes frame is along the orbital motion of the satellite while the z axis points towards the earth. An attitude control gas-jet thruster located on the y-axis malfunctions causing

a certain mass of gas to be impulsively ejected along the z axis. Show that the body components of the angular velocity repeat after every orbit. The spacecraft carries an identical thruster to the one that had malfunctioned at the same location, but with its axis along the x axis. Can this thruster be used to arrest the motion of the angular velocity vector? If not, then explain why not. If yes, then describe when the thruster should be fired and with what impulse in order to arrest the angular velocity motion.

89. A spin-stabilized axisymmetric satellite is equipped with a torque actuator that can only generate torques perpendicular to the axis of symmetry. A malfunction causes the actuator to exert a torque having constant components in the body frame. Show that the body components of angular velocity vary in a periodic fashion. Does the rotational kinetic energy remain constant in this case?

Attitude Control

90. Suppose the body in the problems 81 and 82 represents a satellite spin stabilized about its axis of symmetry. Assume that the satellite carries reaction thrusters that can generate any angular impulse orthogonal to the axis of symmetry. Describe the sequence of angular impulses (magnitudes, directions, and time intervals) needed to rotate the direction of the spin axis from $[0\ 0\ 1]^{\mathrm{T}}$ to $[0\ 1\ 0]^{\mathrm{T}}$ with respect to an inertial frame. (Use problem 82.)

91. An axisymmetric satellite having principal moments of inertia A, A and C carries thrusters that can impart angular impulse orthogonal to the axis of symmetry. The spacecraft is initially spin stabilized about the axis of symmetry with an angular momentum magnitude H. It is desired to change the orientation (with respect to an inertial frame) of the spin axis by an angle $0 < \theta < \pi$ by using two impulsive maneuvers. Find the total angular impulse magnitude required and the shortest time in which the reorientation can be achieved. Find the total angular impulse and shortest time required if the same orientation change is to be achieved in two steps of angular displacement $\theta/2$ each. Which requires more angular impulse, the one-step or the two-step reorientation? Which requires more time? (Use problem 82.)

92. The minimum impulse bit of an attitude control thruster is the smallest amount of linear momentum that the thruster can impart in one pulse. Two pairs of thrusters are mounted diametrically opposite on a cylindrical satellite for controlling the rotation about the symmetry axis. The thrusters are fired in pairs, one pair for clockwise rotation and one for anticlockwise. The thrusters fire whenever the satellite strays from within an angular window of width θ. If the thrusters are at a distance R from the axis, and each thruster has a minimum impulse bit equal to p, then show that the satellite consumes fuel at a rate proportional to $p^2R/I\theta$ where I is the moment of inertia about the axis of symmetry.