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Spaceflight Mechanics

Prob 56: Unit vectors i and j of a frame B have components $[0 \;\; 1/\sqrt{2} \;\; 1/\sqrt{2}]^T$ and $[1/\sqrt{3} \;\; -1/\sqrt{3} \;\; 1/\sqrt{3}]^T$, respectively, in frame I. Find the rotation matrix that can be used to transform the components of a vector in B to its components in I.

Soln: Let R be the rotation matrix which converts components in B to components in I

$$(\bar{V})_I = R(\bar{V})_B \; ; \quad \text{where} \quad R = \begin{bmatrix} i\cdot l & j\cdot l & k\cdot l \\ i\cdot m & j\cdot m & k\cdot m \\ i\cdot n & j\cdot n & k\cdot n \end{bmatrix}$$

l, m, n are the unit vectors of I
i, j, k are the unit vectors of B

Given:

$$i\cdot l = 0 \; ; \quad i\cdot m = 1/\sqrt{2} \; ; \quad i\cdot n = 1/\sqrt{2}$$

$$j\cdot l = 1/\sqrt{3} \; ; \quad j\cdot m = -1/\sqrt{3} \; ; \quad j\cdot n = 1/\sqrt{3}$$

let $\quad k\cdot l = a \; ; \quad k\cdot m = b \; ; \quad k\cdot n = c$

$$\Rightarrow \quad R = \begin{bmatrix} 0 & 1/\sqrt{3} & a \\ 1/\sqrt{2} & -1/\sqrt{3} & b \\ 1/\sqrt{2} & 1/\sqrt{3} & c \end{bmatrix}$$

But we know that R is a special orthogonal matrix ~~Sa~~

$$R^T R = I$$

$$\det R = 1$$

$$R^T R = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ a & b & c \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{3} & a \\ 1/\sqrt{2} & -1/\sqrt{3} & b \\ 1/\sqrt{2} & 1/\sqrt{3} & c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{b+c}{\sqrt{2}} \\ 0 & 1 & \frac{a-b+c}{\sqrt{3}} \\ \frac{b+c}{\sqrt{2}} & \frac{a-b+c}{\sqrt{3}} & a^2+b^2+c^2 \end{bmatrix} = I$$

$$\Rightarrow \quad b+c=0 \quad \Rightarrow \quad b=-c$$

$$a-b+c=0 \quad \Rightarrow \quad a=-2c$$

$$a^2+b^2+c^2=1 \quad \Rightarrow \quad c = \pm 1/\sqrt{6}$$

$$\det R = \begin{vmatrix} 0 & 1/\sqrt{3} & -2c \\ 1/\sqrt{2} & -1/\sqrt{3} & -c \\ 1/\sqrt{2} & 1/\sqrt{3} & c \end{vmatrix} = \frac{-6c}{\sqrt{6}} = 1$$

Only possible when $c = -1/\sqrt{6}$

$c = +1/\sqrt{6}$ is not a solution

$$\therefore \quad R = \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$