Random variables & density functions.

A random variable is an outcome of an experiment that is subject to random effects. A random variable takes values in ~~the~~ a sample space $\Omega \subseteq \mathbb{R}$.

Ex. The # of black cards out of 4 cards drawn at random from a shuffled deck of cards. $\Omega = \{0, 1, 2, 3, 4\}$.
— discrete random variable.

Ex. Error in a single bearing measurement.
Sample space $\Omega = [0, 360°]$.
— ct. random variable.

A random variable is described by its probability density function, pdf.

The pdf of a scalar random variable x is a function $f_x(x)$ s.t. for any subset A of the sample space Ω,

$$\Pr(x \in A) = \int_A f_x(x)\,dx$$

Since $\Pr(x \in \Omega) = 1$, $\int_\Omega f_x(x)\,dx = 1$.

Since $\Pr(x \in A) \geq 0 \;\forall A \subseteq \Omega$, $f_x(x) \geq 0 \;\forall x \in \Omega$.

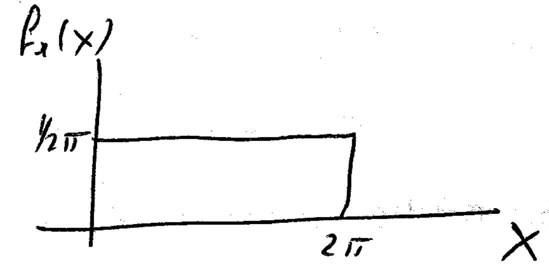
Ex. wheel of fortune $\Omega = [0, 2\pi]$

x - position where the wheel stops. If all stopping positions are equally likely, then

$$Pr\left(x \in [0, \pi/6]\right) = \frac{\pi/6}{2\pi} = \frac{1}{12}.$$

$$\therefore f_x(x) = \frac{1}{2\pi}, \quad X \in [0, 2\pi].$$

$$= 0 \qquad \text{otherwise.}$$



This is a uniform density function, & the random variable is said to be uniformly distributed.

The probability density function (PDF) of a random variable x is a function such that

$$F_x(a) = Pr.\,(x \le a)$$

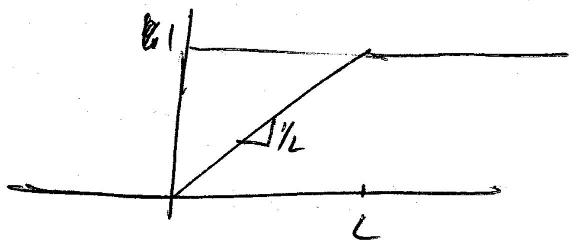
$$= \int_{-\infty}^{a} f_x(x)\,dX$$

Since $f_x(x) \geq 0$, $F_x(x)$ is increasing.

Also $\lim\limits_{X \to -\infty} F_x(x) = 0$, $\lim\limits_{X \to \infty} F_x(x) = 1$.

& $Pr.\,(a \le x \le b) = F_x(b) - F_x(a)$.

The PDF for a uniformly distributed r. v. is.



The mean (also expectation, expected value) of a random variable x is

$$\bar{x} = E(x) = \int_{-\infty}^{\infty} x f_x(x)\, dx. \qquad \text{(like c.g)}$$

f like density

The practical significance is that the arithmetic mean of the values of x obtained in repeated trials converges to $\bar{x}$ with ~~high probability~~. almost surely.

If ~~g is a fun~~ $g: \mathbb{R} \to \mathbb{R}$ is a function, & x is a random variable, then $y = g(x)$ is a random variable.

To find the PDF of y, assume g is increasing. Then g is invertible., g^{-1} is increasing.

$$\therefore F_y(y) = \Pr.(y \le y) = \Pr.(x \le g^{-1}(y))$$
$$= F_x(g^{-1}(y))$$

$$\therefore \text{pdf } f_y(y) = \frac{d}{dy} F_y(y) = \frac{d}{dy} F_x(g^{-1}(y)) = f_x(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$
$$= f_x(g^{-1}(y))$$

$$\therefore \bar{y} = \int_{-\infty}^{\infty} y\, f_y(y)\, dy = \int_{-\infty}^{\infty} g(x) \cdot \frac{f_x(g^{-1}(y))}{\frac{dy}{dx}(g^{-1}(y))}\, dy.$$

put $y = g(x)$, $dy = \frac{dy}{dx}(x)\, dx.$

$$\therefore \bar{y} = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx.$$

More generally, $E(g(x)) = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx.$

The variance of x is given by

$$\sigma_x^2 = E\left[(x-\bar{x})^2\right] = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_x(x)\, dx \quad \boxed{\text{like m.i}}$$

— mean square error.

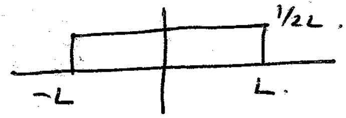
σ_x is the standard deviation, & ~~is the r.m~~ can be thought of as the r.m.s. value of the deviation from the mean.

Ex. If x is a r.v. & a_1, a_2 are constants, then $y = a_1 + a_2 x$ is a r.v with

mean $\bar{y} = a_1 + a_2 \bar{x}$

& variance $\sigma_y^2 = a_2^2 \sigma_x^2$.

Ex. Variance of a uniform distribution.



$$= \int_{-L}^{L} \frac{x^2}{2L}\, dx = \left.\frac{x^3}{6L}\right|_{-L}^{L} = \frac{L^2}{3}$$

Variance increases as spread increases.