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Estimation:

Suppose two sensors yield measurements x_1 & x_2 for a scalar position coordinate x. Both measurements are subject to random errors δx_1 & δx_2. Thus

$$x_1 = x + \delta x_1$$
$$x_2 = x + \delta x_2 .$$

Is it possible to combine the two measurements in a way that yields a better estimate then each sensor by itself?

Suppose we try a linear combination. Then the position estimate is

$$\hat{x} = x + \delta x = a_1 \overbrace{(x + \delta x_1)}^{\text{meas.}} + a_2 \overbrace{(x + \delta x_2)}^{\text{meas.}}$$

where δx is the estimation error.

Assume δx_1, δx_2 ~~as~~ have zero mean, & the covariance matrix

$$\begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_1 x_2} & \sigma_{x_2}^2 \end{bmatrix}.$$

~~Then~~, We would like the estimate to be an unbiased, that is, we want δx to be zero mean.

$\therefore$ we want $a_1 + a_2 = 1$.

~~The~~ $\therefore$ $\delta x = a_1 \delta x_1 + a_2 \delta x_2$.

$\therefore$ variance of the estimate is

$$E[(\delta x - 0)^2] = E[a_1^2 \delta x_1^2 + 2a_1 a_2 \delta x_1 \delta x_2 + a_2^2 \delta x_2^2]$$
$$= a_1^2 \sigma_{x_1}^2 + 2a_1 a_2 \sigma_{x_1 x_2} + a_2^2 \sigma_{x_2}^2 .$$

We wish to choose a_1 & a_2 s.t σ_x^2 is minimized.

Substituting for a_1 yields

$$\sigma_x^2 = (1-a_2)^2\sigma_{x_1}^2 + 2a_2(1-a_2)\,\sigma_{x_1,x_2} + a_2^2\sigma_{x_2}^2.$$

Setting $\frac{d}{da_2} = 0$ yields

$$a_2 = \frac{\sigma_{x_1}^2 - \sigma_{x_1,x_2}}{\sigma_{x_1}^2 + \sigma_{x_2}^2 - 2\sigma_{x_1,x_2}}.$$

Note: $\frac{d^2}{da_2^2} = \sigma_{x_1}^2 - 2\sigma_{x_1x_2} + \sigma_{x_2}^2 > 0$ since P_{xx} is +ve definite.

$\therefore$ The value of a_2 yields a minimum.

$$a_1 = 1 - a_2 = \frac{\sigma_{x_2}^2 - \sigma_{x_1x_2}}{\sigma_{x_1}^2 + \sigma_{x_2}^2 - 2\sigma_{x_1x_2}}.$$

$$(\sigma_x^2)_{min} = a_1^2\sigma_{x_1}^2 + 2a_1a_2\sigma_{x_1x_2} + a_2^2\sigma_{x_2}^2.$$

$$= \frac{\sigma_{x_1}^2\sigma_{x_2}^2 - \sigma_{x_1x_2}^2}{(\sigma_{x_1}^2 + \sigma_{x_2}^2 - 2\sigma_{x_1x_2})}.$$

~~Spl case: Uncorrelated sensor errors.~~ ~~$\sigma_{x_1x_2} = 0$~~

~~In this case~~ **Note:** $(\sigma_x^2)_{min} - \sigma_{x_i}^2 = -\dfrac{(\sigma_{x_i}^2 - \sigma_{x_1x_2})^2}{(\sigma_{x_1}^2 + \sigma_{x_2}^2 - 2\sigma_{x_1x_2})} \leq 0.$ for $i = 1, 2$.

$\therefore$ The optimal estimate has less variance than either measurement by itself.

<u>Spl. case:</u> Uncorrelated sensor noise : $\sigma_{x_1 x_2} = 0$

In this case, $a_1 = \dfrac{\sigma_{x_2}^2}{\sigma_{x_1}^2 + \sigma_{x_2}^2}$. $a_2 = \dfrac{\sigma_{x_1}^2}{\sigma_{x_1}^2 + \sigma_{x_2}^2}$,

Note that the ~~unreliable~~ less ~~sensor~~ reliable sensor ~~has~~ is weighted less.

Also $(\sigma_{xx}^2)_{min} = \dfrac{\sigma_{x_1}^2 \sigma_{x_2}^2}{\sigma_{x_1}^2 + \sigma_{x_2}^2}$, i.e $\dfrac{1}{(\sigma_{xx}^2)_{min}} = \dfrac{1}{\sigma_{x_1}^2} + \dfrac{1}{\sigma_{x_2}^2}$,

so that $(\sigma_{xx}^2)_{min} \leq \sigma_{x_i}^2$, $i = 1, 2$.

The estimate of x obtained in this way is called the maximum likelihood estimate (MLE), the minimum variance unbiased estimate, or the least mean squares estimate. (LMSE) Or Best Linear Unbiased Estimate (BLUE)

<u>Note:</u> If $\sigma_{x_1}^2 = \sigma_{x_2}^2$, then $a_1 = a_2 = \frac{1}{2}$. The arithmetic mean gives the best estimate.

<u>Example:</u> Dead reckoning combined with position fixing

Suppose a craft ~~is~~ sails along the x axis ~~with~~ at an uncertain speed, starting from an uncertain initial position.

actual $\rightarrow x(0) = x_i + \delta x_i$, δx_i - zero mean, variance σ_x^2

Speed $V = v_i + \delta v_i$, δv_i " " σ_v^2

actual measured error

Bearing measurement at time T.

Dead reckoning position at time T

$x_{DR} + \delta x_{DR} = x_i + \delta x_i + V_i T + \delta v_i T$, $x_{DR} = x_i + v_i T$

$$\therefore \sigma_{DR}^2 = \sigma_x^2 + 2T\sigma_{xv} + T^2\sigma_v^2.$$

Uncertainty grows with time.

At time T, a position fix x_F is obtained,

Actual position at time $T = x_F + \delta x_F$.

δx_F – error in position fix, zero mean, variance σ_F^2

Assuming the two errors to be uncorrelated, the minimum variance estimate of posn is

$$\hat{x} = \frac{\sigma_F^2}{\sigma_x^2 + 2T\sigma_{xv} + T^2\sigma_v^2 + \sigma_F^2} x_{DR}$$

$$+ \frac{\sigma_x^2 + 2T\sigma_{xv} + T^2\sigma_v^2}{\sigma_x^2 + 2T\sigma_{xv} + T^2\sigma_v^2 + \sigma_F^2} x_F.$$

The estimate is better than each.

- ~~idea behind~~ idea applies whenever multiple sensors provide overlapping information.
- requires propagating uncertainty till next available fix.
- idea behind Kalman filter.
  - GPS & INS