Strategic Missiles.

Eg. Ballistic missiles : Trajectory consists of 3 phases

a) Lift off & boost phase (~ few minutes)

b) Free suborbital flight.

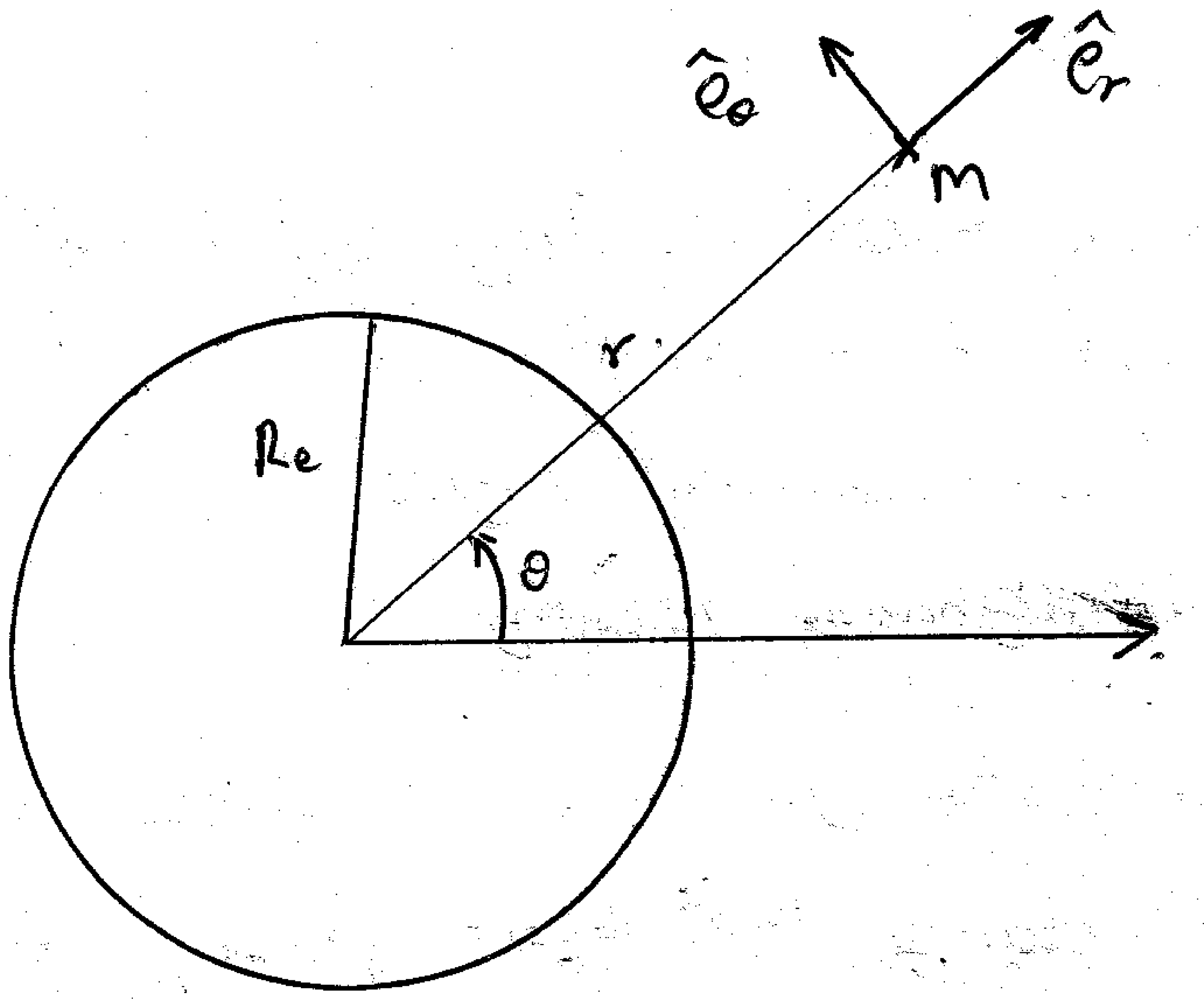
c) Aerodynamic reentry

~~Guidance~~ Ballistic guidance involves determining acceleration to be applied during the boost phase, & determining the cut off conditions. Guidance in subsequent phases is difficult due to the large velocities involved.

The second kind of strategic missiles are strategic interceptors designed to intercept incoming ballistic missiles.

Because of the large distances & durations involved, a flat earth gravity model is inadequate for analysing missile motion. ~~In this~~ We will consider the motion of a missile, assuming the earth to be perfectly spherical and homogeneous, & neglecting atmospheric drag. Under these assumptions, the motion of the missile is given by Newton's inverse square law of gravitation.

We will begin by deriving & solving the ~~necess~~ relevant equations.



Note that as r & θ change, the local unit vectors change according to $\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$, $\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$.

We have $\bar{r} = r\hat{e}_r$

$$\therefore \dot{\bar{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\ddot{\bar{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

By Newton's law,

$$m\ddot{\bar{r}} = -\frac{GMm}{r^2}\hat{e}_r$$

$$\therefore \ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \qquad \mu = GM$$

$$\text{while} \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

We recognize $r^2\dot{\theta}$ to be the angular momentum of the missile about the center of the earth. It remains constant at a value determined by the initial conditions. Let h denote the constant value of the angular momentum. Thus

$$h = r^2\dot{\theta},$$

$\therefore$ The equation for r becomes

$$\ddot{r} - \frac{h^2}{r^3} + \frac{\mu}{r^2} = 0.$$

Note that if the 'r' equation ~~can be~~ is solved, then θ can be obtained by integrating $\dot{\theta} = \frac{h}{r^2}$.

To solve the 'r' equation, let $u = \frac{1}{r}$.

Then $\dot{u} = \dot{\theta} \cdot \frac{du}{d\theta} = \frac{h}{r^2} \frac{du}{d\theta} = hu^2 \frac{du}{d\theta}$.

On the other hand, $\dot{u} = -\frac{1}{r^2}\dot{r} = -u^2 \dot{r}$.

On comparing, we conclude that $\dot{r} = -h\frac{du}{d\theta}$

$$\therefore \ddot{r} = \frac{d}{dt}\left[-h\frac{du}{d\theta}\right] = \frac{d\theta}{dt} \cdot \frac{d}{d\theta}\left[-h\frac{du}{d\theta}\right].$$

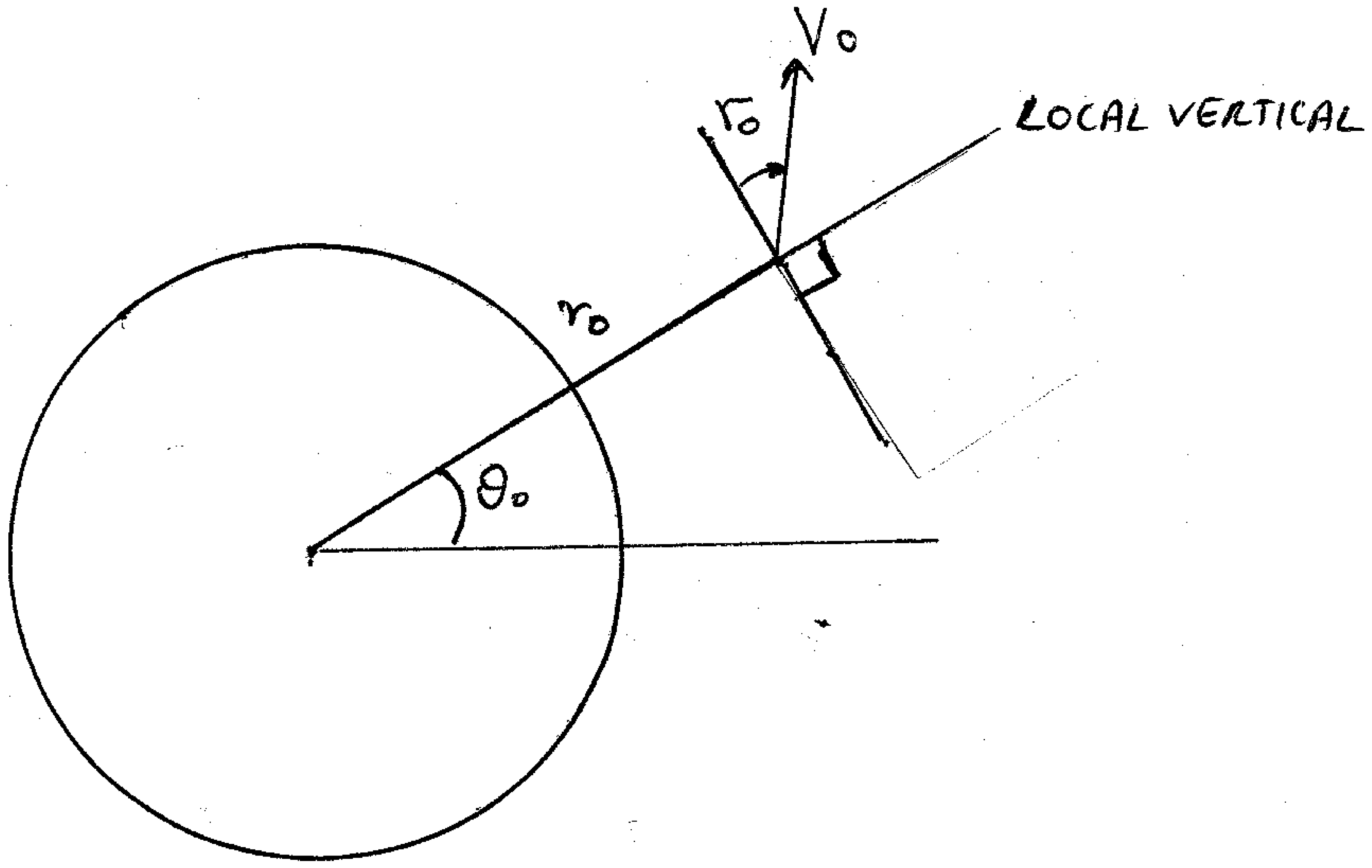
$$= \frac{h}{r^2} \cdot (-h) \cdot \frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Hence the equation for r becomes

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 + \mu u^2 = 0,$$

which yields $$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}.$$

To ~~sta~~ solve the 'u' equation, we need to determine initial values for u & $\frac{du}{d\theta}$. The initial conditions on u depend on the cutoff conditions for the missile, expressed in terms of the cutoff radius r_0, cutoff speed V_0 & flight path angle γ_0



In the figure above, $r(0) = r_0$, $\dot{r}(0) = V_0 \sin\gamma_0$.

$h = r_0 V_0 \cos\gamma_0$. (+ve in the figure).

$$\therefore\ u(0) = \frac{1}{r_0} \qquad \frac{du}{d\theta}(0) = -\frac{1}{h}\dot{r}(0) = \frac{-V_0 \sin\gamma_0}{r_0 V_0 \cos\gamma_0}$$

$$= -\frac{\tan\gamma_0}{r_0}$$

A general solution for u is

$$u(\theta) = A\sin\theta + B\cos\theta + \frac{\mu}{h^2}.$$

where $u(0) = B + \dfrac{\mu}{h^2} = \dfrac{1}{r_0} \Rightarrow B = \dfrac{1}{r_0} - \dfrac{\mu}{h^2}$

& $u'(0) = A = -\dfrac{\tan\gamma_0}{r_0}$

$$\therefore\ u(\theta) = -\frac{\tan\gamma_0}{r_0}\sin\theta + \left(\frac{1}{r_0} - \frac{\mu}{h^2}\right)\cos\theta + \frac{\mu}{h^2}$$

$$\therefore\ r(\theta) = \frac{h^2/\mu}{1 - \dfrac{h^2}{\mu r_0}\tan\gamma_0 \sin\theta + \left(\dfrac{h^2}{\mu r_0} - 1\right)\cos\theta}.$$