AE 695 – State Space Methods

End-semester Exam, Friday, 24/11/06, 2:30pm-5:30pm, Open Notes, 50 marks

1. Let $\mathcal{S}$ denote the set of solutions on the interval $[0, \infty)$ of the homogeneous differential equation $y + 3\ddot{y} - 4y = 0$.

   (a) Show that $\mathcal{S}$ is a vector space over $\mathbb{R}$. (2)

   (b) Let v_1, v_2 and v_3 be functions defined by $v_1(t) = e^t$, $v_2(t) = e^{-2t}$ and $v_3(t) = te^{-2t}$. Show that the set $\mathcal{V} = \{v_1, v_2, v_3\}$ is a basis for $\mathcal{S}$, that is, $\mathcal{V}$ is contained in $\mathcal{S}$, is linearly independent, and spans $\mathcal{S}$. (Hint: Solutions to linear differential equations are unique.) (6)

   (c) Define an operator $\mathcal{L}$ by $\mathcal{L}(v) = \dot{v}$. Show that $\mathcal{L}$ maps elements of $\mathcal{S}$ into $\mathcal{S}$ (that is, $\mathcal{L}(v)$ is a solution whenever v is). (2)

   (d) Find the representation of $\mathcal{L}$ in the basis $\mathcal{V}$. (3)

   (e) What are the kernel and range of $\mathcal{L}$ when considered as an operator on $\mathcal{S}$? (1)

2. Suppose the system $\dot{x} = Ax$, $y = Cx$ is observable. Show that, if there exists a symmetric positive-definite matrix P of the same size as A such that $A^{\mathrm{T}}P + PA = -C^{\mathrm{T}}C$, then the system is asymptotically stable. (5)

3. For what values of the parameters $p \in \mathbb{R}$ and $q \in \mathbb{R}$ do the following set of equations possess i) no solution ii) at least one solution iii) exactly one solution and iv) more than one solution? In the case where the equations possess more than one solution, write down the set of all solutions. (8)

$$\begin{bmatrix} 1 & 1 & p \\ 1 & p & 1 \\ p & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ q \end{bmatrix}$$

4. Which of the following are vector spaces over $\mathbb{R}$? Explain briefly. a) Set of polynomials with integer coefficients b) Set of real-valued functions defined on $\mathbb{R}$ that have the value 0 at 1 c) Set of all $x \in \mathbb{R}^3$ satisfying $x_1 x_2 = 0$. (3)

5. Suppose $\mathcal{U} = \{x \in \mathbb{R}^3 : x_1 = x_3\}$ and $\mathcal{V} = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$. Find $\mathcal{U} + \mathcal{V}$. (3)

6. Write down all possible Jordan forms of matrices having the following characteristic and minimal polynomials: $(s-2)^4(s-3)^2$, $(s-2)^2(s-3)^2$. (3)

7. Find the reachable and unobservable subspaces of the following system. Use your answer to guess the poles of the system. Can you write down a minimal realization for the system by observation? (8)

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix} x. \end{aligned}$$

8. Suppose $P \in \mathbb{R}^{n\times n}$ satisfies $A^{\mathrm{T}}P + PA = -Q$, where $Q \in \mathbb{R}^{n\times n}$ is symmetric positive definite, and $A \in \mathbb{R}^{n\times n}$. Calculate $\frac{d}{dt}e^{A^{\mathrm{T}}t}Pe^{At}$. Use you answer to show that, for every t, the matrix $e^{A^{\mathrm{T}}t}Pe^{At} - P$ is negative definite. (4)

9. Is the quadratic function $V(x_1, x_2, x_3) = 4x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 4x_1x_3 + 4x_2x_3$ positive definite? (2)