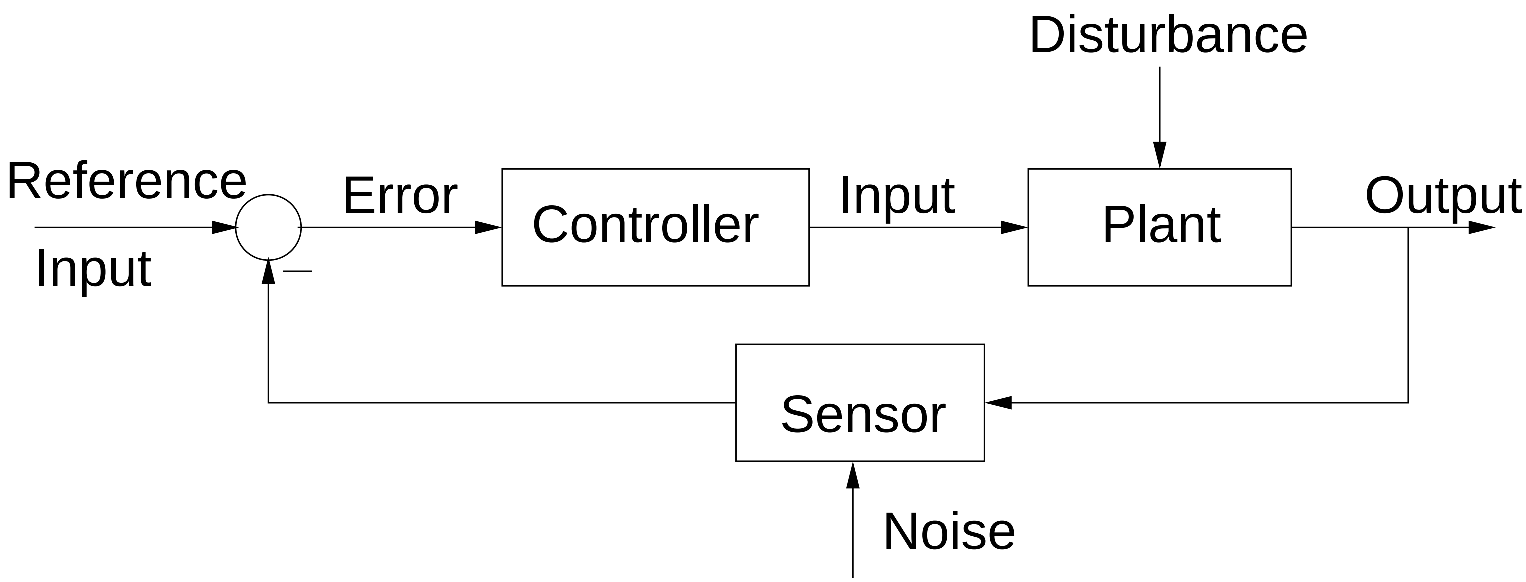
A Typical Feedback System



Typical Control Objectives

- Uncontrolled system (plant) may not behave satisfactorily

  $\Rightarrow$ Design a control system that yields satisfactory behavior for the controlled system

- Typical properties desired of a controlled system
  - **Stability**
    - Input-output stability: Bounded inputs should give bounded outputs
    - Internal stability: All internal variables remain bounded in the absence of inputs
  - **Tracking:** (Output − Input)$\rightarrow 0$ as $t \rightarrow \infty$
    - Regulation: Output $\rightarrow 0$ as $t \rightarrow \infty$
  - **Disturbance/Noise Rejection:** Satisfactory performance in the presence of plant disturbances and measurement noise
  - **Robustness:** Satisfactory performance inspite of unmodelled dynamics and parameter uncertainty/change

Review of Continuous-Time Systems

- All signals are analog signals
- A linear, time invariant, single-input-single-output (SISO) system is typically described by

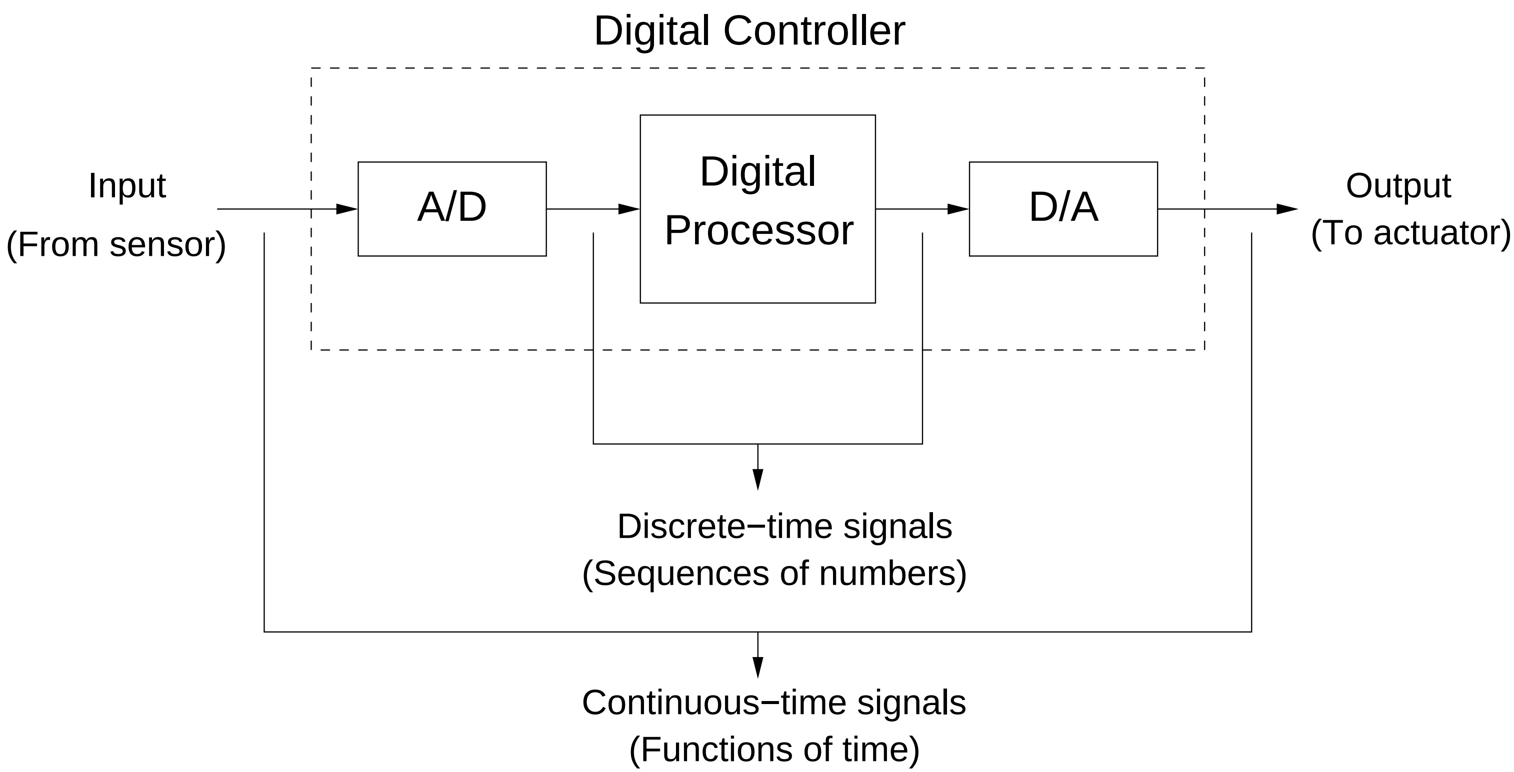
$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{(n)} y = b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_m u$$

- Solution $=$ initial condition response $+$ input response
- Input response $= u*$ impulse response (convolution)
- Transfer function $= \mathcal{L}$(impulse response)
- $\mathcal{L}(y) = \mathrm{T.F.} \times \mathcal{L}(u)$ for zero initial conditions
- Transient response decided by poles and zero; poles decide stability
- Frequency response analysis: Harmonic Analysis, Bode, Nyquist

An Overview of Control Activities

- ANALYSIS:
  - Relate system theoretic properties to system behaviour.
    Eg. Poles and stability
  - Need analysis tools, eg. Routh-Hurwitz test
- CONTROLLER DESIGN:
  - Translate specs to system properties and design a controller (control law) that assigns these properties to the controlled system
    Eg. Pole placement controller
  - Need design tools, eg. pole placement technique
- IMPLEMENTATION:
  - Sensors, actuators, filters, processors

A Digital Controller



Discrete-Time Signals

- Sequence $\{u(k)\}_{k=0}^{\infty}$ of real numbers
- A real-valued function $k \mapsto u(k)$ of integers
- Right-sided sequence (signal) $u(0), u(1), u(2), \ldots$
- Two-sided sequence (signal) $\ldots, u(-2), u(-1), u(0), u(1), u(2), \ldots$

Operators on Discrete-Time Signals

- Identity operator 1
- Shift or unit delay operator $\mathcal{S}$

$$\begin{aligned}(\mathcal{S}u)(k) &= u(k-1),\ k \geq 1 \\ &= 0 \quad k \leq 1\end{aligned}$$

- Unit advance operator $\mathcal{S}^{-1}$

$$(\mathcal{S}^{-1}u)(k) = u(k+1),\ k \geq 0$$

- Difference operator

$$\Delta u(k) = u(k) - u(k-1) = (u - \mathcal{S}u)(k)$$

$$\Delta = 1 - \mathcal{S},\ \mathcal{S} = 1 - \Delta$$

Some Basic Discrete-Time Signals

- Unit pulse/impulse signal

$$\begin{aligned} \delta(k) &= 1, \ k = 0, \\ &= 0, \ k > 0, \end{aligned}$$

- Unit step signal

$$s(k) = 1, \ k \geq 0$$

$$\Delta s = \delta$$

- Harmonic signals

$$u(k) = \sin(k\theta)$$

- Exponential signals

$$u(k) = r^k$$

- Harmonic signals with exponential amplitudes

$$u(k) = r^k \sin(k\theta) = \text{Re } (re^{\jmath\theta})^k$$

Linear Difference Equations

$$y(k) + a_1 y(k-1) + \ldots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \ldots + b_m u(k-m)$$

- In terms of the shift operator

$$y(k) + a_1 \mathcal{S} y(k) + \ldots + a_n \mathcal{S}^n y(k) = b_0 u(k) + b_1 \mathcal{S} u(k) + \ldots + b_m \mathcal{S}^m u(k)$$

$$D(\mathcal{S}) y = N(\mathcal{S}) u$$

- Auto-Regressive Moving Averages (ARMA) model
- Causal: Output independent of future input
  - Strictly causal if output depends only the past inputs
- Shift invariant (time invariant)
  - Shifted input $\mathcal{S}u$ produces shifted output $\mathcal{S}y$
- Linear (Superposition + Homogeneity)
- To solve need n initial conditions + input

An Example

- To numerically compute

$$y(t) = \int_0^t u(\tau)d\tau$$

$$\dot{y}(t) = u(t),\ y(0) = 0$$

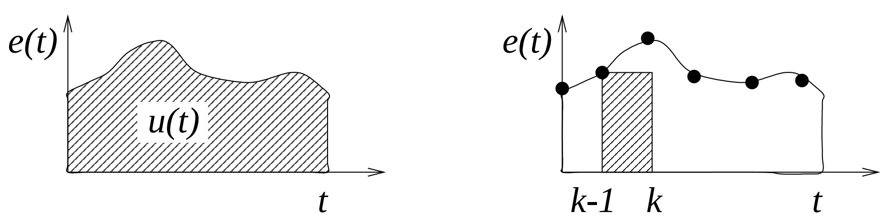
- At instants $0, T, 2T, \ldots, kT, \ldots,$

$$y(kT) = y((k-1)T) + \int_{(k-1)T}^{kT} u(\tau)d\tau$$

- Use **forward rectangular rule** to approximate the integral

$$y(kT) = y((k-1)T) + Tu((k-1)T),\ y(0) = 0$$

$$\Delta y = T\mathcal{S}u,\ y(0) = 0$$



Vector Spaces

- A vector space $\mathcal{V}$ is a set
  - whose elements can be added in some manner
  - whose elements can multiplied by scalars in some manner
  - which contains a zero element

  For example:

  - $\mathcal{V}$ = set of all functions of time
  - $\mathcal{V}$ = set of all right-sided sequences

Linear Independence

- A *linear combination* is a finite sum of the form

$$\alpha_1 v_1 + \ldots + \alpha_n v_n$$

- Linear independence — every linear combination involving atleast one nonzero scalar is nonzero

- $\{v_1, v_2, \ldots, v_n\} \subset \mathcal{V}$ forms a basis for $\mathcal{V}$ if
  - v's are linearly independent and
  - every vector in $\mathcal{V}$ is a linear combination of v's

- If $\mathcal{V}$ has a basis of n elements for some n, then $\mathcal{V}$ is n-dimensional, else infinite-dimensional

- A *linear operator* is a linear function $\mathcal{V} \mapsto \mathcal{V}$

Vector Space of Discrete Signals

- The set of all discrete signals is a vector space with

$$(y_1 + y_2)(k) = y_1(k) + y_2(k),\ (\alpha y)(k) = \alpha y(k),\ \text{Zero element } y \equiv 0$$

- $y_1, \ldots, y_n$ are linearly dependent iff $\exists\ \alpha_1, \ldots, \alpha_n$ such that

$$\alpha_1 y_1(k) + \ldots + \alpha_n y_n(k) = 0\ \forall k$$

- For $\lambda_1 \neq \lambda_2$ nonzero real, $\{\lambda_1^k\}$, $\{\lambda_2^k\}$ are linearly independent
- For λ complex, $\{\text{Re } \lambda^k\}$ and $\{\text{Im } \lambda^k\}$ are l. i.
- For $\lambda_1 \neq \lambda_2$ complex, $\{\text{Re } \lambda_{1,2}^k\}$ and $\{\text{Im } \lambda_{1,2}^k\}$ are l. i.
- For λ_1 nonzero real and λ_2 complex, λ_1^k, $\{\text{Re } \lambda_2^k\}$ and $\{\text{Im } \lambda_2^k\}$ are l. i.
- No finite basis possible
- Linear operators

$$\mathcal{S},\ D(\mathcal{S}),\ \Delta,\ \widehat{D}(\Delta)$$

Homogeneous Linear Difference Equations

$$y(k) + a_1 y(k-1) + \ldots + a_n y(k-n) = 0$$

$$D(\mathcal{S})y = 0$$

$$y(-1), y(-2), \ldots, y(-n)$$

- Zero initial conditions imply solution is zero
- The set of all solutions is a vector space since

$$y \equiv 0 \text{ is a solution}$$

$$D(\mathcal{S})(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 D(\mathcal{S})y_1 + \alpha_2 D(\mathcal{S})y_2$$

Theorem: The vector space of solutions has dimension n

($a_n \neq 0$)

- To prove, need to find a basis consisting of n solutions

A Basis of Solutions

- **Idea** Solutions $\longleftrightarrow$ sets of initial conditions
  - If w_i form a basis for all initial conditions, the corresponding solutions form a basis for all solutions
- Choose n sets of initial conditions as follows

	-1	-2	-3	$\cdots$	$-n$	
y_1	1	0	0	$\cdots$	0	w_1
y_2	0	1	0	$\cdots$	0	w_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_n	0	0	0	$\cdots$	1	w_n

Claim: $y_1, \ldots, y_n$ form a basis for all solutions

A Basis of Solutions (cont'd)

- $y_1, \ldots, y_n$ are linearly independent

$$\text{If } \alpha_1 y_1(k) + \cdots + \alpha_n y_n(k) = 0 \;\forall\; k, \text{ then } k = -i \Rightarrow \alpha_i = 0$$

- Every solution is a linear combination of $y_1, \ldots, y_n$

  Let y be any solution and consider

$$\overline{y}(k) = y(-1)y_1(k) + \cdots + y(-i)y_i(k) + \cdots + y(-n)y_n(k)$$

  – $\overline{y}$ is a solution satisfying the same initial conditions as y

  – Hence $y = \overline{y}$ is a linear combination of $y_1, \ldots, y_n$

Solution of Linear Difference Equations

- Choose a basis of initial conditions and use corresponding solutions as a basis of solutions
- Try a solution of the form $y(k) = \lambda^k$

$$(\mathcal{S}y)(k) = \lambda^{k-1} = \lambda^{-1}y(k) \text{ for } k \geq 1$$

$$(\mathcal{S}^2 y)(k) = \lambda^{-2}y(k) \text{ for } k \geq 2$$

$$D(\mathcal{S})y(k) = D(\lambda^{-1})y(k) \text{ for } k \geq n$$

- $y(k) = \lambda^k$ is a solution of $D(\mathcal{S})y = 0$ if λ satisfies

$$D(\lambda^{-1}) = 0, \text{ that is}$$

$$1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \ldots + a_n\lambda^{-n} = 0 \Longrightarrow$$

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0$$

- $\lambda^n D(\lambda^{-1})$ = characteristic polynomial

Another Basis of Solutions

- Characteristic polynomial/equation — factor as

$$C(\lambda) = (\lambda - p_1)^{m_1}(\lambda - p_2)^{m_2} \ldots (\lambda - p_l)^{m_l}$$

- The following functions form a basis for the solutions of the LDE
  - For p_i real,
    $\{p_i^k\},\ \{kp_i^k\},\ \{k^2 p_i^k\},\ \ldots,\ \{k^{m_i-1}p_i^k\}$
  - For $p_i = re^{\jmath\theta}$ and $\overline{p}_i$ complex,
    $\{r^k \sin(k\theta)\},\ \{kr^k \sin(k\theta)\},\ \{k^2 r^k \sin(k\theta)\}, \ldots,\ \{k^{m_i-1} r^k \sin(k\theta)\}$
    $\{r^k \cos(k\theta)\},\ \{kr^k \cos(k\theta)\},\ \{k^2 r^k \cos(k\theta)\}, \ldots,\ \{k^{m_i-1} r^k \cos(k\theta)\}$

- Initial conditions determine the constants in the linear combination

Stability of Initial Condition Response

- Real characteristic root p

$\{p^k\}$	decays iff $\lvert p\rvert < 1$	$\{k^j p^k\}$	decays iff $\lvert p\rvert < 1$
	bounded if $\lvert p\rvert = 1$		
	unbounded if $\lvert p\rvert > 1$		unbounded if $\lvert p\rvert \geq 1$

- Complex characteristic root p

$\{\mathrm{Re}\ p^k\},\ \{\mathrm{Im}\ p^k\}$	decay iff $\lvert p\rvert < 1$	$\{\mathrm{Re}\ k^j p^k\},\ \{\mathrm{Im}\ k^j p^k\}$	decay iff $\lvert p\rvert < 1$
	bounded if $\lvert p\rvert = 1$		
	unbounded if $\lvert p\rvert > 1$		unbounded if $\lvert p\rvert \geq 1$

- **Theorem**
  - All solutions are bounded iff all characteristic roots lie in the closed unit disc $\{\lambda : |\lambda| \leq 1\}$ and all roots with unit magnitude are *simple* (unrepeated)
  - All solutions decay iff all characteristic roots lie in the open unit disc $\{\lambda : |\lambda| < 1\}$

Convolution

- Convolution of two right sided sequences u and g is the sequence

$$\begin{aligned}(u*g)(k) &= \sum_{l=0}^{k} u(l)g(k-l) \\ &= u(0)g(k)+u(1)g(k-1)+\cdots+u(k-1)g(1)+u(k)g(0)\end{aligned}$$

$$\boxed{u*g=g*u}$$

$$(\alpha_1 u_1+\alpha_2 u_2)*g=\alpha_1(u_1*g)+\alpha_2(u_2*g)$$

  – For a fixed g, $u*g$ is a linear operator on u

$$\boxed{\mathcal{S}(u*g)=u*\mathcal{S}g=g*\mathcal{S}u}$$

$$\boxed{D(\mathcal{S})(u*g)=u*D(\mathcal{S})g=g*D(\mathcal{S})u}$$

$$\Delta(u*g)=u*\Delta g=g*\Delta u$$

$$\boxed{u*\delta=u}$$

  – $(u*\delta)(k)=u(0)\delta(k)+\ldots+u(k-1)\delta(1)+u(k)\delta(0)=u(k)$

Pulse Response and Input Response

- Pulse response $g =$ zero initial condition response to a unit pulse

$$D(\mathcal{S})g = N(\mathcal{S})\delta,\ 0 = g(-1) = g(-2) = \cdots$$

- **Fact:** The response y of a linear time invariant system to an arbitrary input u under zero initial conditions is given by

$$y = u * g$$

Proof: To show $D(\mathcal{S})y = N(\mathcal{S})u$

$$\begin{aligned} D(\mathcal{S})y &= D(\mathcal{S})(u * g) \\ &= u * D(\mathcal{S})g \\ &= u * N(\mathcal{S})\delta \\ &= N(\mathcal{S})(u * \delta) \\ &= N(\mathcal{S})u \end{aligned}$$

- Step response $= s * g$

Bounded-Input-Bounded-Output (BIBO) Stability

- A system is BIBO stable if the output to every bounded input is bounded
- A sequence y is said to be bounded if there exists M such that $|y(k)| < M, \ \forall \ k$
- For a bounded sequence y, define

$$\|y\| = \sup_{k>0} y(k) = \text{least upper bound of } \{y(k)\}$$

- **Fact:**

  A system is BIBO stable if and only if there exists N such that for every nonzero input u, the corresponding output y satisfies

$$\frac{\|y\|}{\|u\|} < N$$

- **Theorem:**

  A system is BIBO stable iff the input response g is *absolutely summable*, that is,

$$\sum_{k=0}^{\infty} |g(k)| < \infty$$

BIBO Stability and Pulse Response

- Suppose the pulse response is absolutely summable

$$
\begin{aligned}
|y(k)| &= |(u*g)(k)| \le \sum_{l=0}^{k} |u(l)||g(k-l)| \\
&\le \|u\| \sum_{l=0}^{k} |g(k-l)| \le \|u\| \sum_{l=0}^{\infty} |g(l)| < \infty
\end{aligned}
$$

- Suppose the pulse response is not absolutely summable

$$
\begin{aligned}
u_k(l) &= \text{sign } g(k-l) \quad l \le k, \\
&= 0, \qquad\qquad\qquad l > k
\end{aligned}
$$

$$\|u_k\| = 1$$

$$\|y_k\| \ge y_k(k) = (u_k * g)(k) = \sum_{l=0}^{k} |g(k)|$$

$$\|y_k\| \longrightarrow \infty$$

Output Zeroing Inputs

$$D(\mathcal{S})y = N(\mathcal{S})u$$

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_i u(k-i) + \cdots + b_m u(k-m)$$

- An *output zeroing input* — produces no response under under zero initial conditions, that is, satisfies $u * g = 0$
- Must satisfy the difference equation

$$N(\mathcal{S})u = 0, \text{ that is, } b_i u(k-i) + \cdots b_m u(k-m) = 0$$

- Set of all null inputs is a vector space
- A basis can be found from the *characteristic zeros*, solutions of

$$b_i \lambda^{m-i} + b_{i+1}\lambda^{m-i-1} + \cdots + b_m = 0$$

  – Zero z_i of multiplicity m_i contributes

$$\{z_i^k\}, \{k z_i^k\}, \cdots, \{k^{m_i - 1} z_i^k\}$$

- Dimension of this vector space is $m - i$

Impulse Response and Initial Condition Responses

- Let d be the impulse response of the system $D(\mathcal{S})y = u$, that is, $D(\mathcal{S})d = \delta$
- d is an initial condition response of the system $D(\mathcal{S})d = N(\mathcal{S})u$ since $(D(\mathcal{S})d)(k) = 0,\ k > 0$
- d is a linear combination of the initial condition responses corresponding to the characteristic roots

$$d = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n$$

- Impulse response g of $D(\mathcal{S})y = N(\mathcal{S})u$ is $g = N(\mathcal{S})d$

$$g = \alpha_1 N(\mathcal{S})y_1 + \alpha_2 N(\mathcal{S})y_2 + \cdots + \alpha_n N(\mathcal{S})y_n$$

- Characteristic root p_i affects g iff it is not a characteristic zero

$$g(k) = \alpha_1 N(\mathcal{S})p_1^k + \alpha_2 N(\mathcal{S})kp_1^k + \alpha_3 k^2 p_1^k + \alpha_4 N(\mathcal{S})p_2^k$$

  - g decays iff roots $|p| \geq 1$ are also a zeros of equal or greater multiplicity

Roots, Zeros and BIBO Stability

$$\sum_{k=0}^{\infty} |p^k|, \ \sum_{k=0}^{\infty} |kp^k| < \infty$$

$$\Updownarrow$$

$$|p| < 1$$

$$\Updownarrow$$

$$p^k, \ kp^k \text{ decay}$$

- Since g involves p^k, kp^k, g is absolutely summable iff g decays
- System is BIBO stable iff pulse response decays
- **Theorem:**

  System is BIBO stable iff every characteristic root with $|p| \geq 1$ is also a characteristic zero of equal or greater multiplicity

$\mathcal{Z}$ Transform

- The $\mathcal{Z}$ transform of a sequence is a function of the complex variable z

- Given a sequence y, its right sided $\mathcal{Z}$ transform is

$$\mathcal{Z}(y) : \; Y(z) = y(0) + \frac{y(1)}{z} + \frac{y(2)}{z^2} + \frac{y(3)}{z^3} + \cdots = \sum_{k=0}^{\infty} y(k) z^{-k}$$

- $\mathcal{Z}(y)$ is the Laurent expansion of the complex function Y
  - $\mathcal{Z}(y)$ agrees with Y only in the region of convergence of the Laurent series
  - Recall that if $|x| < 1$, then $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$
  - If $|z^{-1}| < 1$, then

$$1 + z^{-1} + z^{-2} + z^{-3} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

- $\mathcal{Z}$ transform is linear: $\mathcal{Z}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 \mathcal{Z}(y_1) + \alpha_2 \mathcal{Z}(y_2)$

$\mathcal{Z}$ Transforms of Some Common Sequences

- Unit pulse δ

$$\mathcal{Z}(\delta) = 1$$

- Unit step s

$$S(z) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}, \quad |z| > 1$$

- Exponential sequence $\{p^k\}$

$$1 + pz^{-1} + p^2 z^{-2} + \cdots = \frac{1}{1 - pz^{-1}} = \frac{z}{z-p}, \quad |pz^{-1}| < 1$$

- Harmonic signal $\{\sin(k\theta)\}$

$$\frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1$$

- Exponentially modulated harmonic signals $\{r^k \sin(k\theta)\}$

$$\frac{rz \sin\theta}{z^2 - 2rz\cos\theta + r^2}, \quad |z| > r$$

Properties of the $\mathcal{Z}$ Transform

- Delay

$$\mathcal{Z}(\mathcal{S}u) = u(-1) + z^{-1}U(z)$$

$$\mathcal{Z}(\mathcal{S}^2u) = \mathcal{S}u(-1) + z^{-1}\mathcal{Z}(\mathcal{S}u) = u(-2) + z^{-1}u(-1) + z^{-2}U(z)$$

$$\mathcal{Z}(\mathcal{S}^nu) = u(-n) + z^{-1}u(-n+1) + \cdots + z^{n-1}u(-1) + z^{-n}U(z)$$

$$\mathcal{Z}(D(\mathcal{S})u) = D(z^{-1})U(z)$$

- Advance

$$\mathcal{Z}(\mathcal{S}^{-1}u) = zU(z) - zu(0),\ \mathcal{Z}(\mathcal{S}^{-2}u) = z^2U(z) - z^2u(0) - zu(1)$$

$$\mathcal{Z}(\mathcal{S}^{-n}u) = z^nU(z) - z^nu(0) - z^{n-1}u(1)\cdots - zu(n-1)$$

- Difference

$$\mathcal{Z}(\Delta u) = \mathcal{Z}(u) - \mathcal{Z}(\mathcal{S}u) = \frac{z-1}{z}U(z) - u(-1)$$

- Convolution

$$\mathcal{Z}(u * g) = G(z)U(z)$$

Properties of the $\mathcal{Z}$ Transform (Contd.)

- Scaling in the complex plane

$$\mathcal{Z}(\{r^k u(k)\}) = U(z/r)$$

- Complex differentiation

$$\mathcal{Z}(\{ku(k)\}) = -z\frac{dU}{dz}(z)$$

- Initial value

$$u(0) = \lim_{z\to\infty} U(z)$$

- Final value theorem

$$\lim_{k\to\infty} u(k) = \lim_{z\to 1}(z-1)U(z)$$

  – provided the limit on the left exists

Transfer Functions

- The *transfer function* G of the system $D(\mathcal{S})y = N(\mathcal{S})u$ is the $\mathcal{Z}$ transform of its pulse response g

$$G(z) = \mathcal{Z}(g)$$

- If y is the input response (zero i.c.) to the input u, then

$$\begin{aligned} y &= (g * u) \\ Y(z) &= G(z)U(z) \\ G(z) &= \frac{Y(z)}{U(z)} \\ \text{transfer function} &= \left.\frac{\mathcal{Z}(\text{output})}{\mathcal{Z}(\text{input})}\right|_{\text{zero initial conditions}} \end{aligned}$$

- To calculate the transfer function of $D(\mathcal{S})y = N(\mathcal{S})u$, take the $\mathcal{Z}$ transform on both sides

$$\mathcal{Z}(D(\mathcal{S})y) = \mathcal{Z}(N(\mathcal{S})u)$$

$$\boxed{G(z) = \frac{Y(z)}{U(z)} = \frac{N(z^{-1})}{D(z^{-1})}}$$

Transfer Functions of Common Operators

- Unit Delay: $y = \mathcal{S}u$

$$Y(z) = \mathcal{Z}(\mathcal{S}u) = u(-1) + z^{-1}U(z), \ \left.\frac{Y(z)}{U(z)}\right|_{\text{zero i.c.}} = z^{-1}$$

$$\text{Pulse Response} = \mathcal{Z}^{-1}(z^{-1}) = \{0, 1, 0, 0, \ldots\}$$

- Unit advance: $y = \mathcal{S}^{-1}u$

$$Y(z) = zU(z) - u(0), \ Y(z)/U(z) = z$$

  – Non causal. $y(k) = u(k+1)$

- Difference operator: $y(k) = u(k) - u(k-1)$

$$D(\lambda) = 1, \ N(\lambda) = 1 - \lambda, \ \text{T.f.} = \frac{N(z^{-1})}{D(z^{-1})} = 1 - z^{-1}$$

$$\text{Impulse Response} = \{1, -1, 0, 0, \ldots\}$$

  – Causal but not strictly causal

Inverse $\mathcal{Z}$ Transform

- Laurent expansion
  - Perform long division for rational $Y(z)$
- Partial fraction expansion followed by look-up table
- Convolution property

$$Y(z) = Y_1(z)Y_2(z)$$

$$\Longrightarrow y = y_1 * y_2$$

- Solve numerically by forming a linear difference equation

$$Y(z) = \frac{N(z^{-1})}{D(z^{-1})}$$

$$\Longrightarrow y = \text{pulse response of } D(\mathcal{S})y = N(\mathcal{S})u$$

Partial Fractions

$$Y(z) = \frac{N(z^{-1})}{D(z^{-1})} = \frac{N(z^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})^2\cdots}$$

- Unrepeated factor $1 - pz^{-1}$ contributes to the expansion

$$\frac{A}{1-pz^{-1}}$$

- Repeated factor $(1 - pz^{-1})^m$ contributes

$$\frac{A_m}{(1-pz^{-1})^m} + \frac{A_{m-1}}{(1-pz^{-1})^{m-1}} + \cdots + \frac{A_1}{(1-pz^{-1})}$$

- Unrepeated quadratic factor $1 - 2rz^{-1}\cos\theta + z^{-2}$ contributes

$$\frac{Az^{-1}+B}{1-2rz^{-1}\cos\theta+z^{-2}}$$

- Repeated quadratic $(1 - 2rz^{-1}\cos\theta + z^{-2})^m$ factor contributes

$$\frac{A_mz^{-1}+B_m}{(1-2rz^{-1}\cos\theta+z^{-2})^m} + \cdots + \frac{A_1z^{-1}+B_1}{1-2rz^{-1}\cos\theta+z^{-2}}$$

- Expand in terms of z^{-1} (not z) in the usual fashion

- Inverse transform each term in the expansion using tables

The $s - z$ Correspondence

- $y(t) = e^{\sigma t}$: $Y(s)$ has a pole at $s = \sigma$
  - $y(kT) = e^{\sigma kT} = (e^{\sigma T})^k = r^k$: $Y(z)$ has a pole at $z = r = e^{\sigma T}$
- $y(t) = e^{\sigma t} \sin \omega t$: $Y(s)$ has a pole at $s = \sigma \pm \imath\omega$
  - $y(kT) = e^{\sigma kT} \sin \omega kT = (e^{\sigma T})^k \sin k(\omega T) = r^k \sin k\theta$: $Y(z)$ has poles at $z = re^{\pm \imath\theta} = e^{\sigma T} e^{\pm \imath\omega T} = e^{(\sigma \pm \imath\omega)T}$
- Suggests the correspondence $z = e^{sT}$ for mapping poles of a s-domain signal to the poles of its sampled sequence in z-domain
- Where should z poles lie to get good transient behaviour $(\zeta, \omega_{\mathrm{n}})$?
  - Locate s poles using s domain experience for desired $\zeta, \omega_{\mathrm{n}}$
  - Map s poles to z poles using $z = e^{sT}$

Jury's Test for Stability

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0, \;\; a_0 > 0$$

– is said to be *Hurwitz* if all roots lie in the OLHP, *Schur* if all roots lie in the OUD

z^n	z^{n-1}	z^{n-2}	$\cdots$	z^2	z	z^0
a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\cdots$	a_2	a_1	a_0
b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_1	b_0	
c_0	c_1	c_2	$\cdots$	c_{n-2}		
c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	c_0		
$\vdots$	$\vdots$	$\vdots$				

$$b_k = \frac{1}{a_0} \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \;\; k = 0, 1, \ldots, n-1, \;\; c_k = \frac{1}{b_0} \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, \;\; k = 0, 1, \ldots, n-2$$

- Stable if $a_0 > 0$, $b_0 > 0$, $c_0 > 0$, $\ldots$

Stability through $s-z$ Transformation

$$z = \frac{1+sT/2}{1-sT/2}, \;\; s = \frac{2}{T}\frac{(z-1)}{(z+1)}$$

$$\text{OLHP} \longleftrightarrow \text{open unit disc}$$

$$\text{imaginary axis} \longleftrightarrow \text{unit circle}$$

- Given a polynomial $p(z)$,

$$p(z) = p\left(\frac{1+sT/2}{1-sT/2}\right) = \frac{n(s)}{d(s)}$$

$$\text{zeros of } p(z) \longleftrightarrow \text{zeros of } n(s)$$

- p is Schur **iff** n is Hurwitz

- Apply Routh-Hurwitz test to $n(s)$

BIBO Stability of Transfer Functions

- A system given by a transfer function $G(z)$ is BIBO stable
  - if and only if the impulse response g is absolutely summable
  - if and only if the impulse response g decays
  - if and only if all poles (after cancellation) of $G(z)$ lie in the interior of the unit disc, the open unit disc (OUD)

    $\{z : |z| < 1\}$
- We call a transfer function stable if all its poles lie in the OUD

Step Response

$$Y(z) = G(z)(1 - z^{-1})^{-1}$$

- Bounded if (after pole-zero cancellation)
  - all poles of $G(z)$ lie in $\{z : |z| \leq 1\}$
  - no repeated poles on the unit circle
  - no pole at $z = 1$
- Approaches a limit if
  - all poles of $G(z)$ lie in $\{z : |z| < 1\}$

$$\text{Steady state value } = \lim_{k \to \infty} y(k) = \lim_{z \to 1} (z - 1)Y(z) = G(1)$$

- Decays to zero if
  - all poles of $G(z)$ lie in $\{z : |z| < 1\}$ and $z = 1$ is a zero, that is, $G(1) = 0$
- For *asymptotic tracking* of a step input, need stability + $G(1) = 1$
- For *asymptotic rejection* of a step disturbance, need stability + $G(1) = 0$

Harmonic Response of Stable Transfer Functions

$$u(k) = \sin(k\omega T)$$

$$U(z) = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} = \frac{z^{-1} \sin \omega T}{(1 - e^{\jmath\omega T} z^{-1})(1 - e^{-\jmath\omega T} z^{-1})}$$

$$Y(z) = G(z)U(z) = \frac{a_1}{1 - e^{\jmath\omega T} z^{-1}} + \frac{a_2}{1 - e^{-\jmath\omega T} z^{-1}} + \frac{b_1}{1 - p_1 z^{-1}} + \cdots$$

$$a_1 = G(z)U(z)(1 - e^{\jmath\omega T} z^{-1})\Big|_{z=e^{\jmath\omega T}} = \frac{1}{2\jmath} G(e^{\jmath\omega T}) = \frac{1}{2\jmath} r e^{\jmath\phi}$$

$$a_2 = \bar{a}_1 = -\frac{1}{2\jmath} r e^{-\jmath\phi}$$

$$Y_{\text{ss}}(z) = \frac{a_1}{1 - e^{\jmath\omega T} z^{-1}} + \frac{a_2}{1 - e^{-\jmath\omega T} z^{-1}}$$

$$y_{\text{ss}}(k) = a_1 (e^{\jmath\omega T})^k + a_2 (e^{-\jmath\omega T})^k = \text{Im } r e^{\jmath\phi} (e^{\jmath\omega T})^k$$

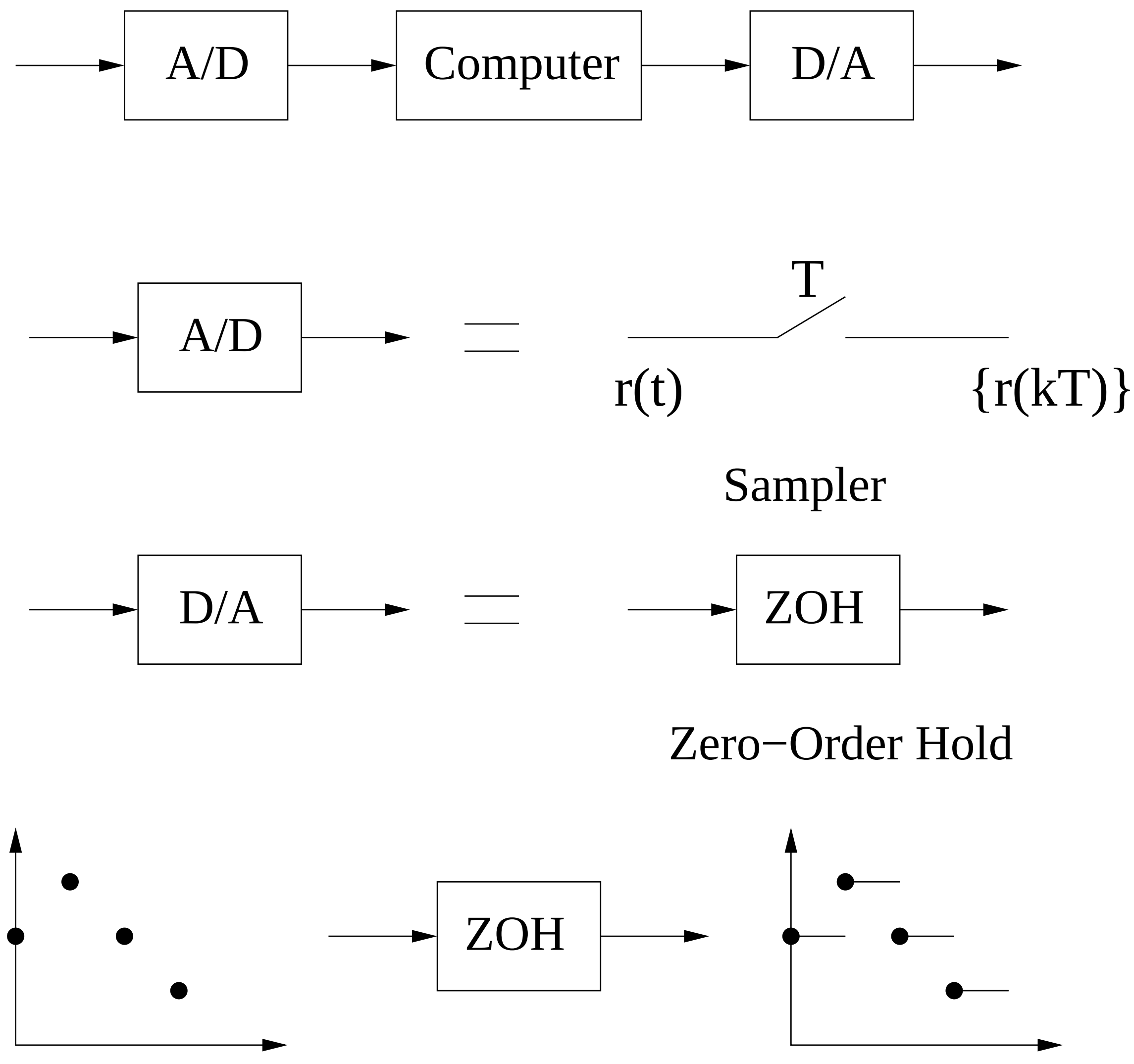
$$\boxed{y_{\text{ss}}(k) = r \sin(k\omega T + \phi),\ r = |G(e^{\jmath\omega T})|,\ \phi = \angle G(e^{\jmath\omega T})}$$

- Amplification at ω is $|G(e^{\jmath\omega T})|$, phase difference is $\angle G(e^{\jmath\omega T})$

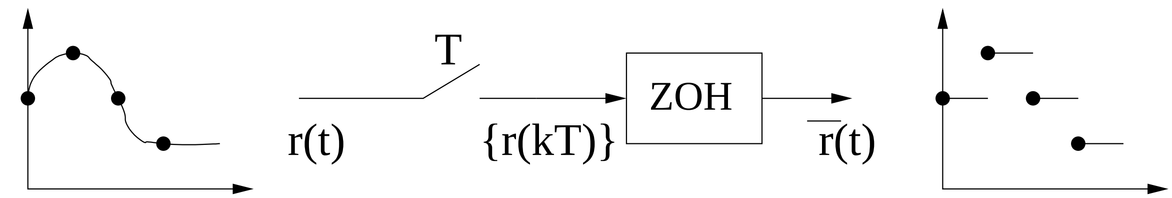
- Frequency response is periodic in frequency

Digital – Analog Conversion

– Same response at ω and $\omega + \omega_s$



Analysis of Sample-and-Hold Operation



- Let $\overline{s}(t)$ = unit step function, $\overline{s}(t-kT)$ = unit step function delayed by kT

$$\begin{aligned}\overline{r}(t) &= r(0)[\overline{s}(t)-\overline{s}(t-T)]+r(T)[\overline{s}(t-T)-\overline{s}(t-2T)]+\cdots \\ &= \sum_{k=0}^{\infty} r(kT)[\overline{s}(t-kT)-\overline{s}(t-kT-T)]\end{aligned}$$

$$\mathcal{L}(\overline{s}(t)) = s^{-1},\ \mathcal{L}(\overline{s}(t-kT)) = s^{-1}e^{-skT}$$

$$\overline{R}(s) = \underbrace{\left[\sum_{k=0}^{\infty} r(kT)(e^{-sT})^k\right]}_{R^*(s)} \underbrace{\left[\frac{1-e^{-sT}}{s}\right]}_{G_{\text{ZOH}}(s)}$$

$$\mathcal{Z}(R(s)) \stackrel{\text{def}}{=} \mathcal{Z}(\text{sampled sequence of } r(t))$$

$$R^*(s) = \mathcal{Z}(R(s))\big|_{z=e^{sT}}$$

Sampler as an Impulse Modulator

- Let $\overline{\delta}(t)$ = unit impulse in continuous time

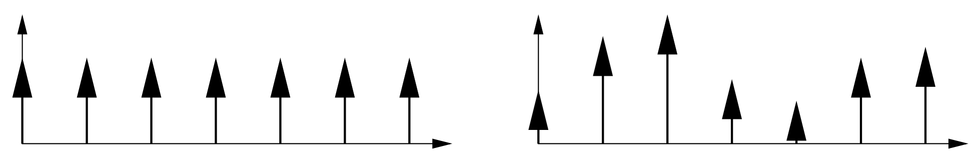
$$\mathcal{L}(\overline{\delta}(t)) = 1,\ \mathcal{L}(\overline{\delta}(t-kT)) = e^{-skT}$$

$$R^*(s) = \mathcal{L}\left(\sum_{k=0}^{\infty} r(kT)\overline{\delta}(t-kT)\right) = \mathcal{L}(r^*(t))$$

- Define $\delta_T(t) = \sum_{k=0}^{\infty} \overline{\delta}(t-kT)$
  - δ_T is an *impulse train*

$$r^*(t) = r(t)\delta_T(t)$$

  - r^* is a *modulated* impulse train

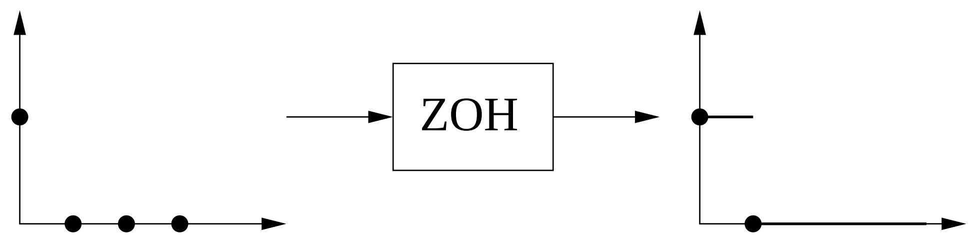


$\delta_T(t)$ $r^*(t)$

  - Ideal sampler = impulse modulator

An Ideal ZOH

$$
\begin{aligned}
G_{\text{ZOH}}(s) &= \frac{1}{s} - \frac{e^{-sT}}{s} \\
&= \mathcal{L}[\overline{s}(t) - \overline{s}(t-T)] \\
&= \mathcal{L}[\text{“impulse}'' \text{ response of ZOH}]
\end{aligned}
$$



- NOTE: No transfer function possible for a ZOH

Frequency Domain Analysis of an Impulse Train

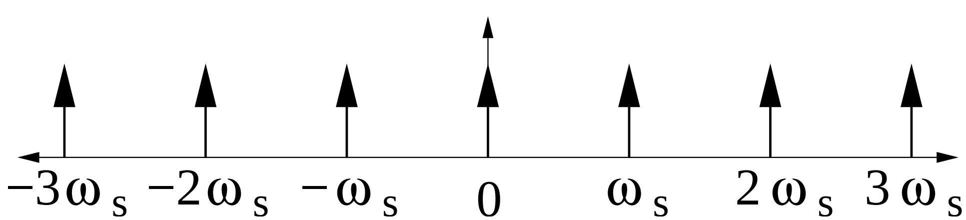
- $\delta_T(t)$ is a periodic function $\Longrightarrow$ expand in a Fourier series

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T}$$

$$\begin{aligned} c_n &= \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-j2\pi nt/T} dt \\ &= \frac{1}{T} \end{aligned}$$

$$\delta_T = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_s t}$$

  - Fourier transform of $\delta_T(t)$

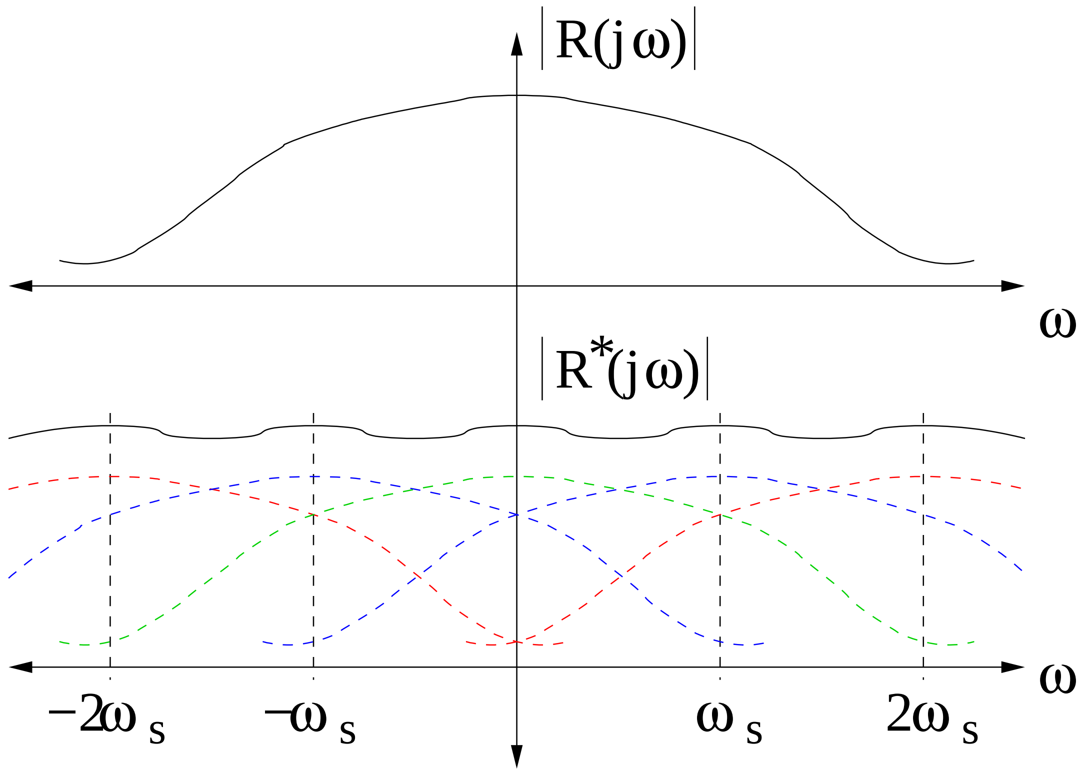


Frequency Domain Analysis of a Modulated Impulse Train

$$\begin{aligned}
R^*(s) &= \mathcal{L}(r^*(t)) = \mathcal{L}(r(t)\delta_T(t)) \\
&= \frac{1}{T}\int_0^\infty r(t)\delta_T(t)e^{-st}dt \\
&= \frac{1}{T}\int_0^\infty r(t)\sum_{n=-\infty}^{\infty} e^{\jmath n\omega_\mathrm{s}t}e^{-st}dt \\
&\stackrel{?}{=} \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_0^\infty r(t)e^{-(s-\jmath\omega_\mathrm{s}n)t}dt \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s-\jmath\omega_\mathrm{s}n)
\end{aligned}$$

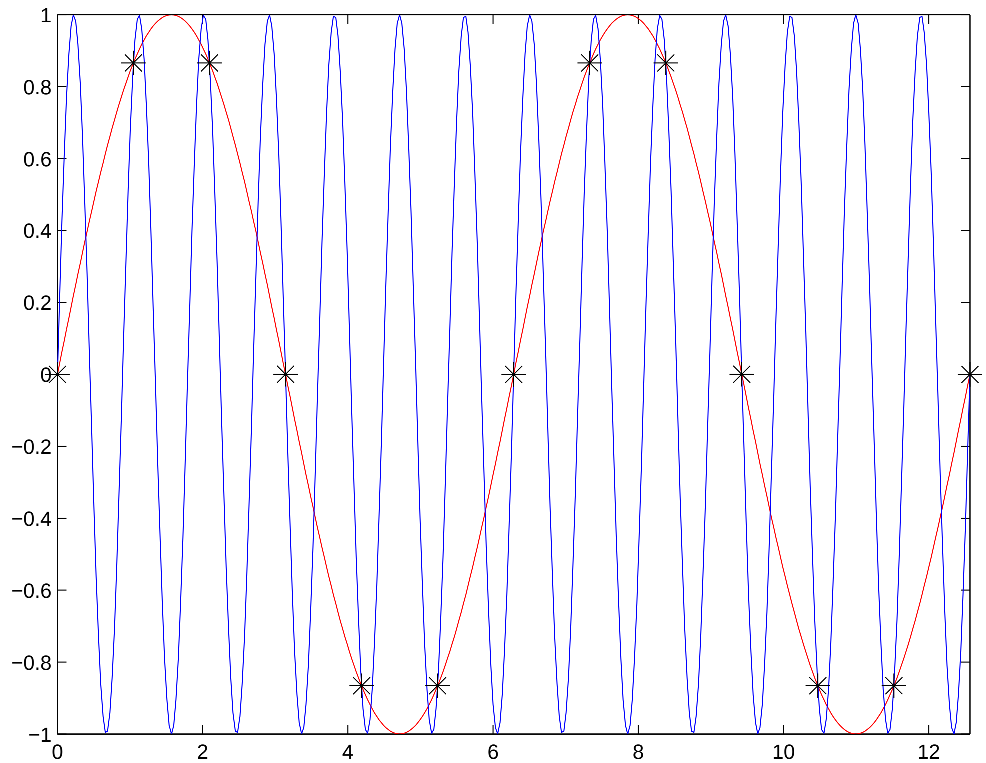
- Fourier transform $R^*(\jmath\omega)$ is periodic in ω with period ω_s
- $R^*(\jmath\omega)$ obtained by superimposing scaled copies of $R(\jmath\omega)$ shifted by multiples of ω_s

Aliasing



- Contributions at ω due to $R(\jmath\omega)$, $R(\jmath\omega \pm n\jmath\omega_{\mathrm{s}})$
- Frequencies $\omega \pm n\omega_{\mathrm{s}}$ aliases of ω, show up at ω after sampling

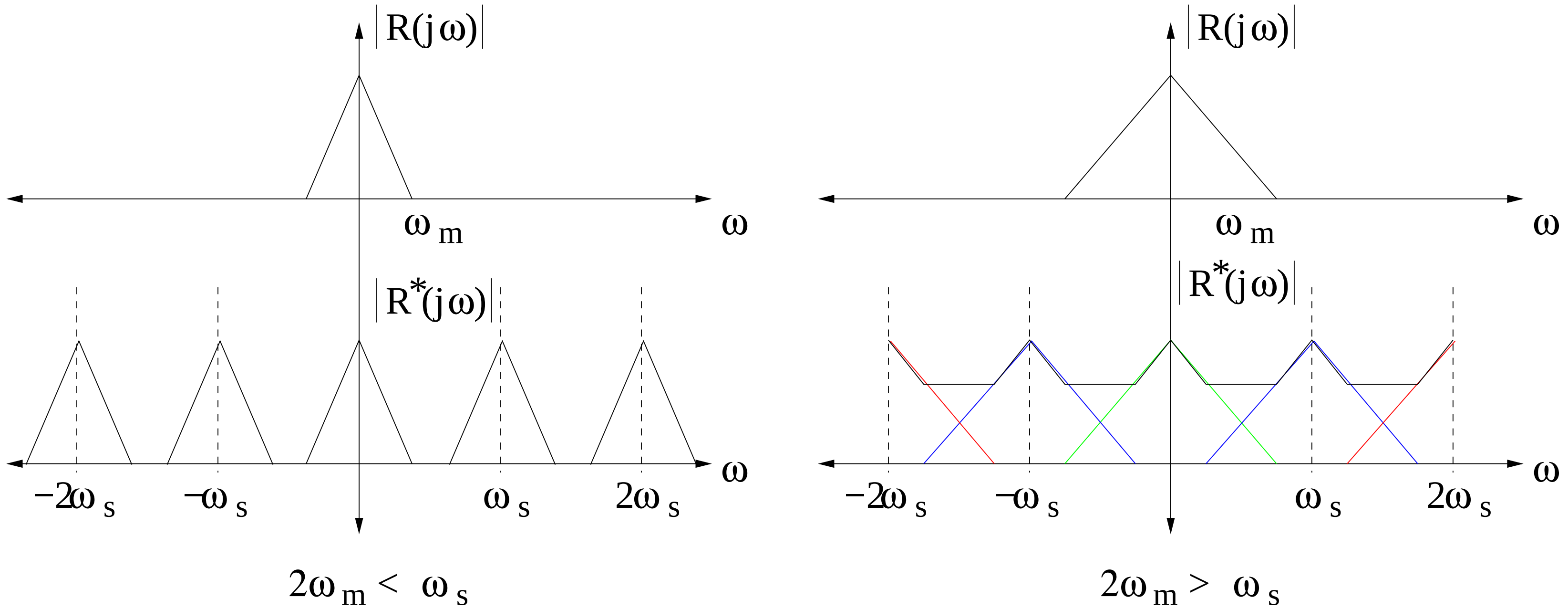
An Example of Aliasing



$$y_1(t) = \sin t,\ \ y_2(t) = \sin 7t,\ \ \omega_{\mathrm{s}} = 6,\ \ T = \pi/3$$

$$y_1(kT) = y_2(kT) = \sin kT$$

Anti-Aliasing



- Nyquist's/Shannon's Sampling Theorem: A signal can be recovered from its samples if the sampling frequency is more than twice the highest frequency in the signal
- To minimise the effect of aliasing, sampling is preceded by a low-pass antialias filter
  - Eliminates frequencies above the Nyquist frequency

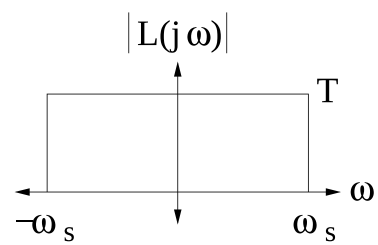
Signal Reconstruction from Samples

- Possible if signal is band limited and $\omega_{\rm m} < \omega_{\rm s}/2$
- To recover $R(\jmath\omega)$ from $R^*(\jmath\omega)$, need a filter Lsuch that

$$R(\jmath\omega) = L(\jmath\omega)R^*(\jmath\omega)$$

$$R^*(\jmath\omega) = \frac{1}{T}R(\jmath\omega) + \underbrace{\frac{1}{T}\sum_{n=-\infty,n\neq 0}^{\infty} R(\jmath\omega - n\jmath\omega_{\rm s})}_{\text{frequencies}>\omega_{\rm s}/2}$$

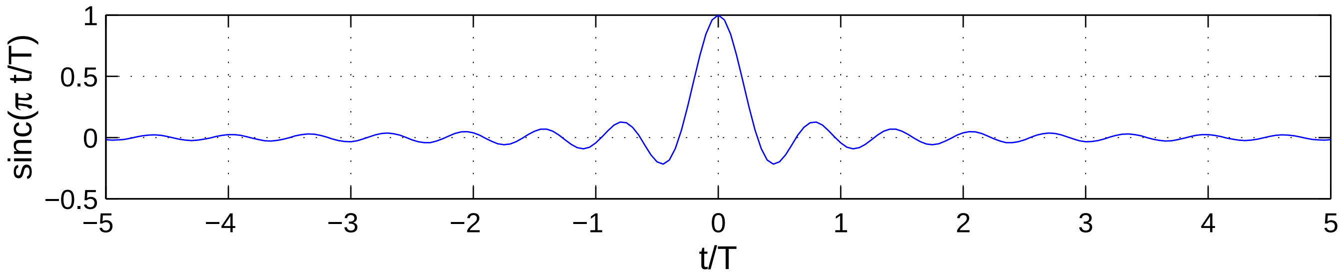
$$\begin{aligned} |L(\jmath\omega)| &= T, \ \omega \in [-\omega_{\rm s}/2, \omega_{\rm s}/2], \\ &= 0, \ \text{elsewhere} \\ \angle L(\jmath\omega) &= 0 \ \text{everywhere} \end{aligned}$$



Impulse Response of an Ideal Low-Pass Filter

- Inverse Fourier transform of $L(\jmath\omega)$

$$
\begin{aligned}
l(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} L(\jmath\omega)e^{\jmath\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_\mathrm{s}/2}^{\omega_\mathrm{s}/2} T e^{\jmath\omega t}d\omega \\
&= \frac{\sin(\omega_\mathrm{s}t/2)}{\omega_\mathrm{s}t/2} = \mathrm{sinc}(\omega_\mathrm{s}t/2)
\end{aligned}
$$



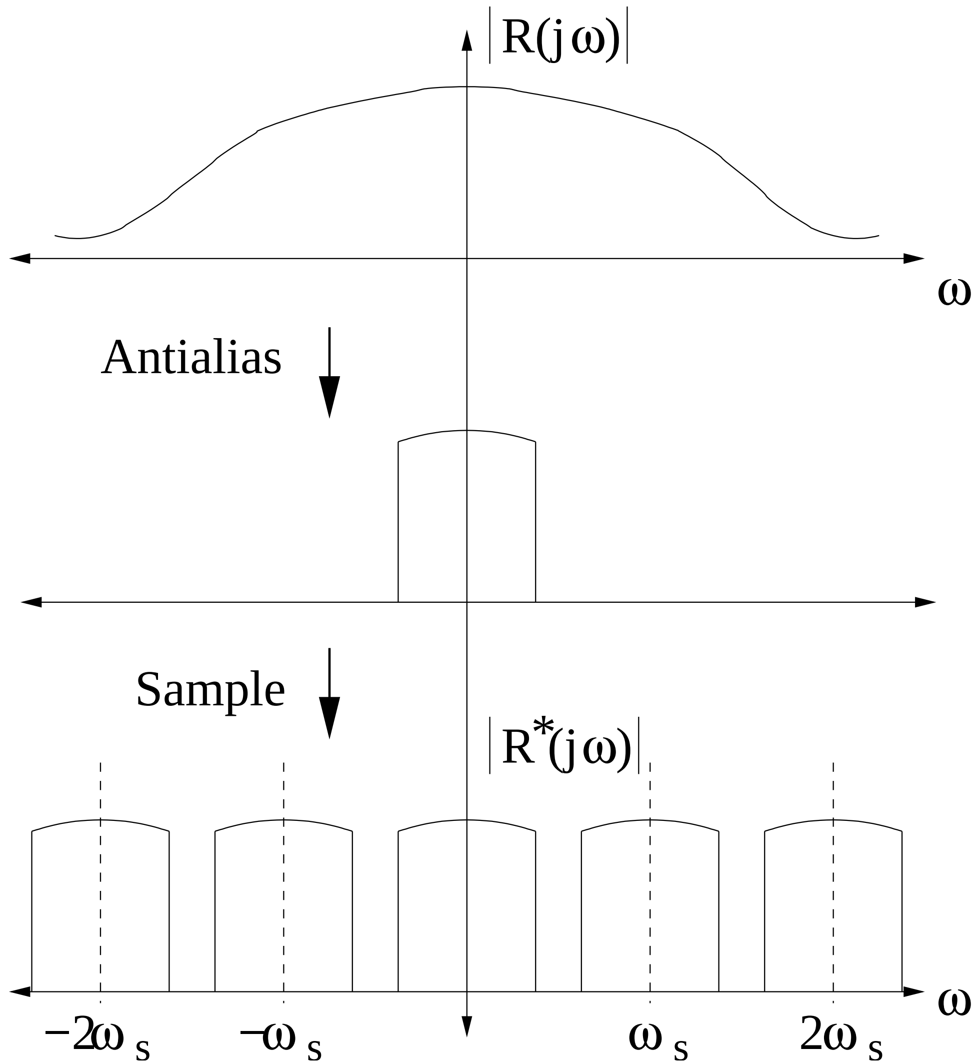
- Note: L is a noncausal filter

Reconstruction Using a Low-Pass Filter

$$
\begin{aligned}
r(t) &= (l * r^*)(t) \\
&= \int_{-\infty}^{\infty} r(\tau)\delta_T(\tau)l(t-\tau)d\tau \\
&= \sum_{k=-\infty}^{\infty} r(kT)\mathrm{sinc}(\omega_{\mathrm{s}}(t-kT)/2)
\end{aligned}
$$

- RHS is the unique band limited signal that has
  - $\omega_{\mathrm{m}} < \omega_{\mathrm{s}}/2$
  - Same samples as r
- Reconstruction is noncausal — present value depends on future samples
- Cannot be implemented online, can be used for offline reconstruction

Antialias Filtering



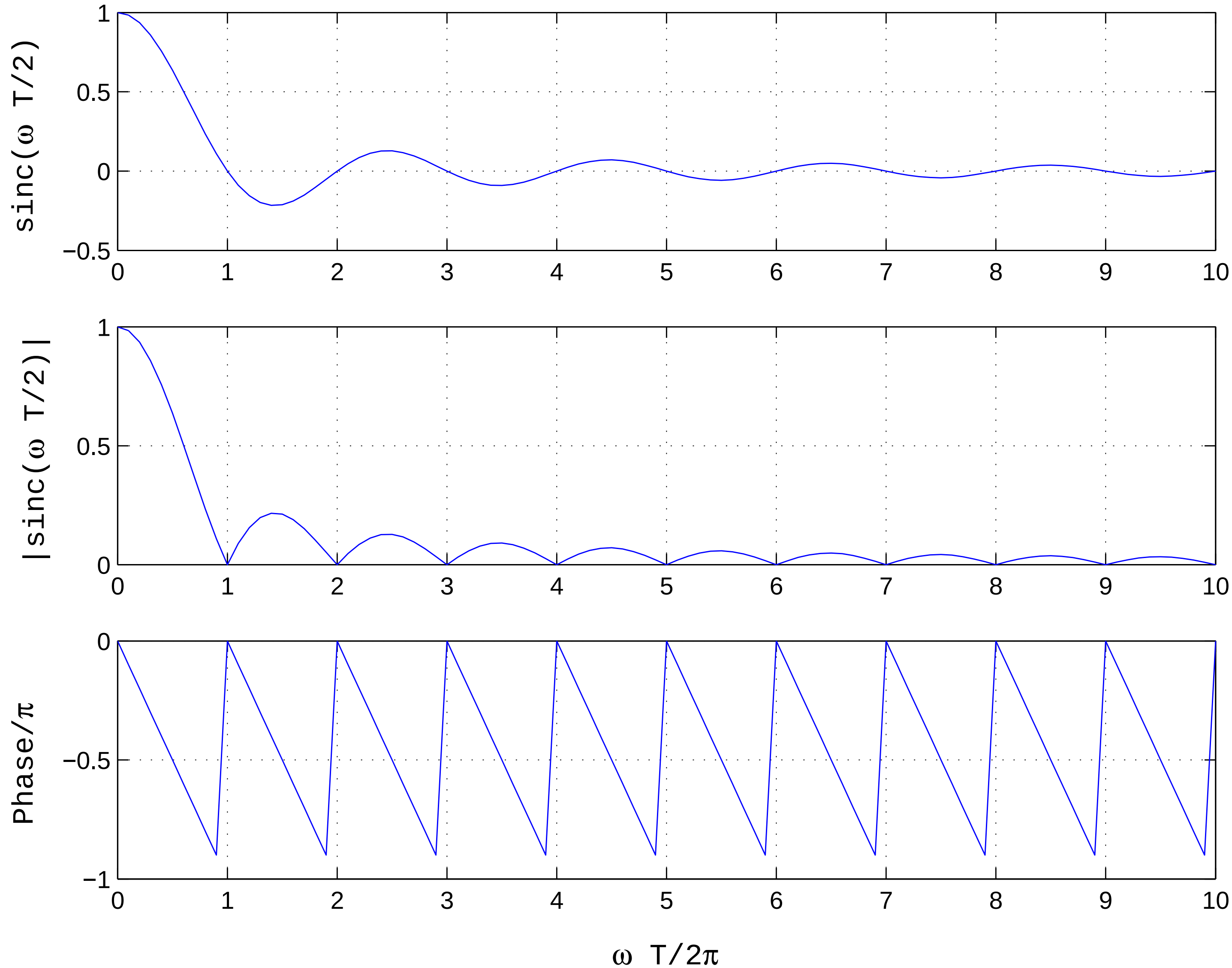
Frequency Domain Analysis of Zero-Order Hold

$$
\begin{aligned}
G_{\text{ZOH}}(\jmath\omega) &= \frac{1 - e^{-\jmath\omega T}}{\jmath\omega} \\
&= T e^{-\jmath\omega T/2} \frac{\sin(\omega T/2)}{(\omega T/2)} \\
&= T e^{-\jmath\omega T/2} \text{sinc}(\omega T/2)
\end{aligned}
$$

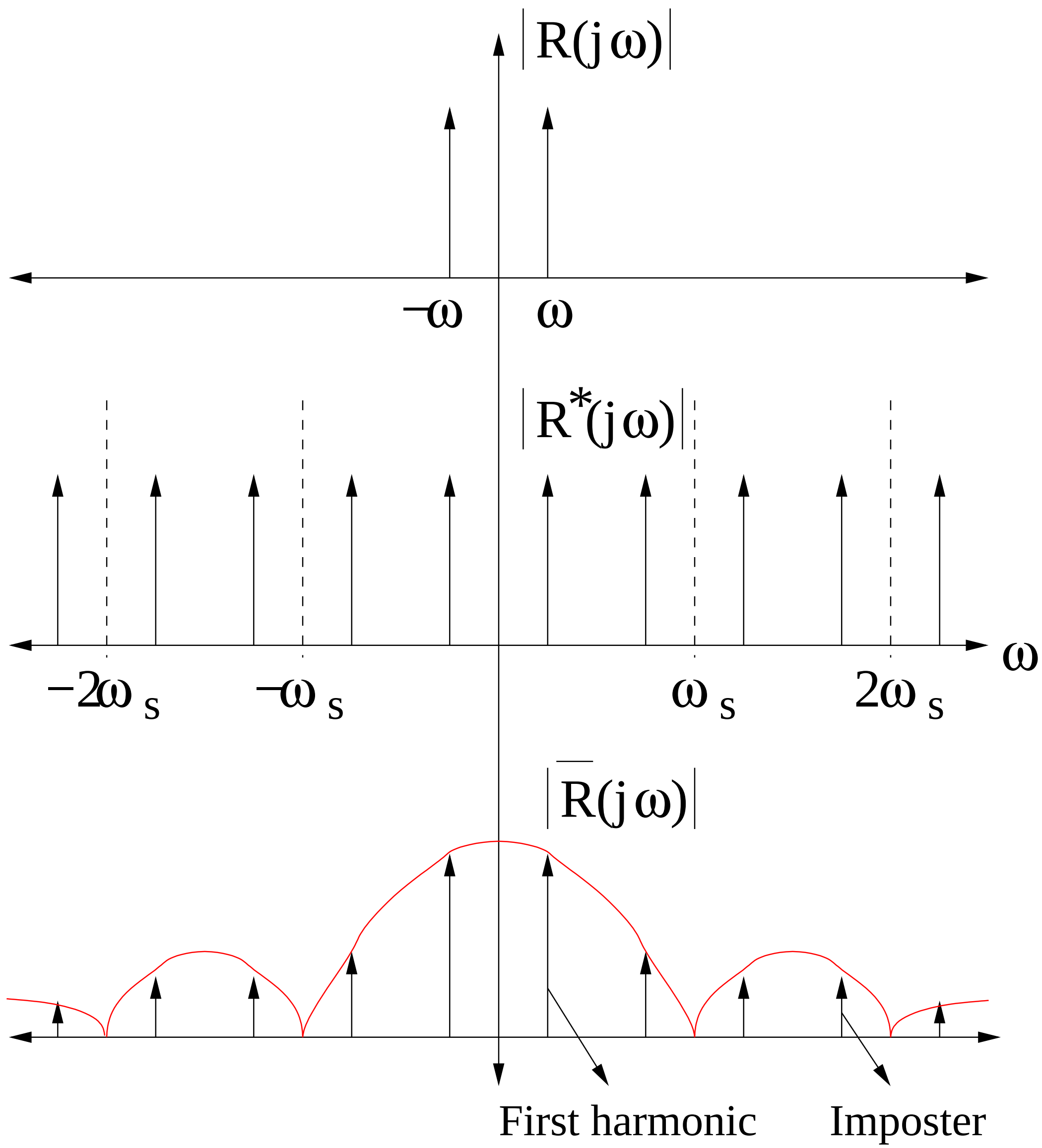
$$
\begin{aligned}
\text{sinc} x &= \frac{\sin x}{x}, \quad x \neq 0, \\
&= 1, \qquad x = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Magnitude} : |G_{\text{ZOH}}(\jmath\omega)| &= T|\text{sinc}(\omega T/2)| \\
\text{Phase} : \angle G_{\text{ZOH}}(\jmath\omega) &= -\frac{\omega T}{2} + \pi \text{ at every sign change of } \text{sinc}
\end{aligned}
$$

Frequency Response of G_{ZOH}



Harmonic Response of Sample and Hold

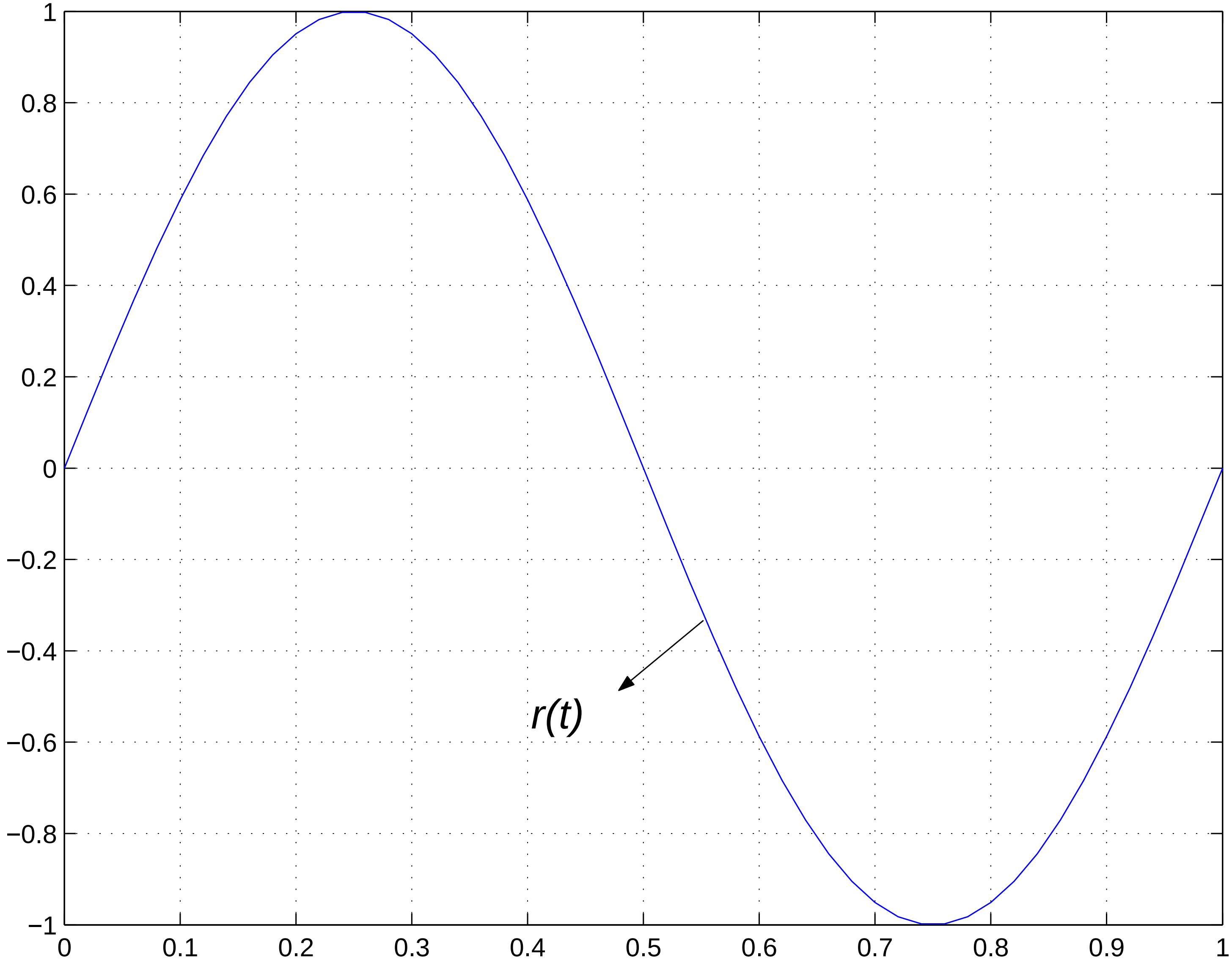


- Input sinusoid of frequency $\omega < \omega_s/2$
- First harmonic of output has

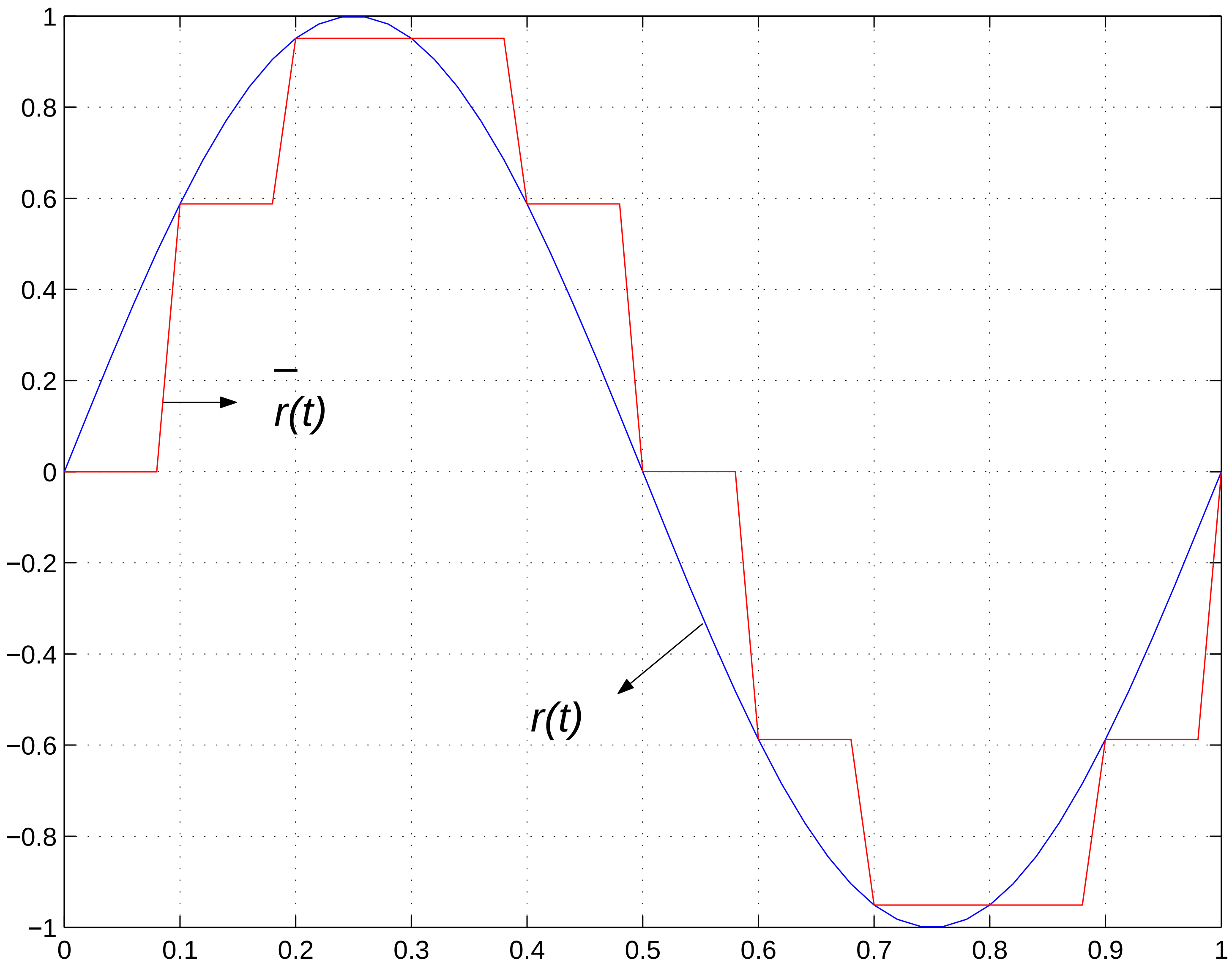
  magnitude : $\mathrm{sinc}(\omega T/2)$

  phase : $-\omega T/2$
- First harmonic is $\mathrm{sinc}(\omega T/2)\sin\omega(t - T/2)$

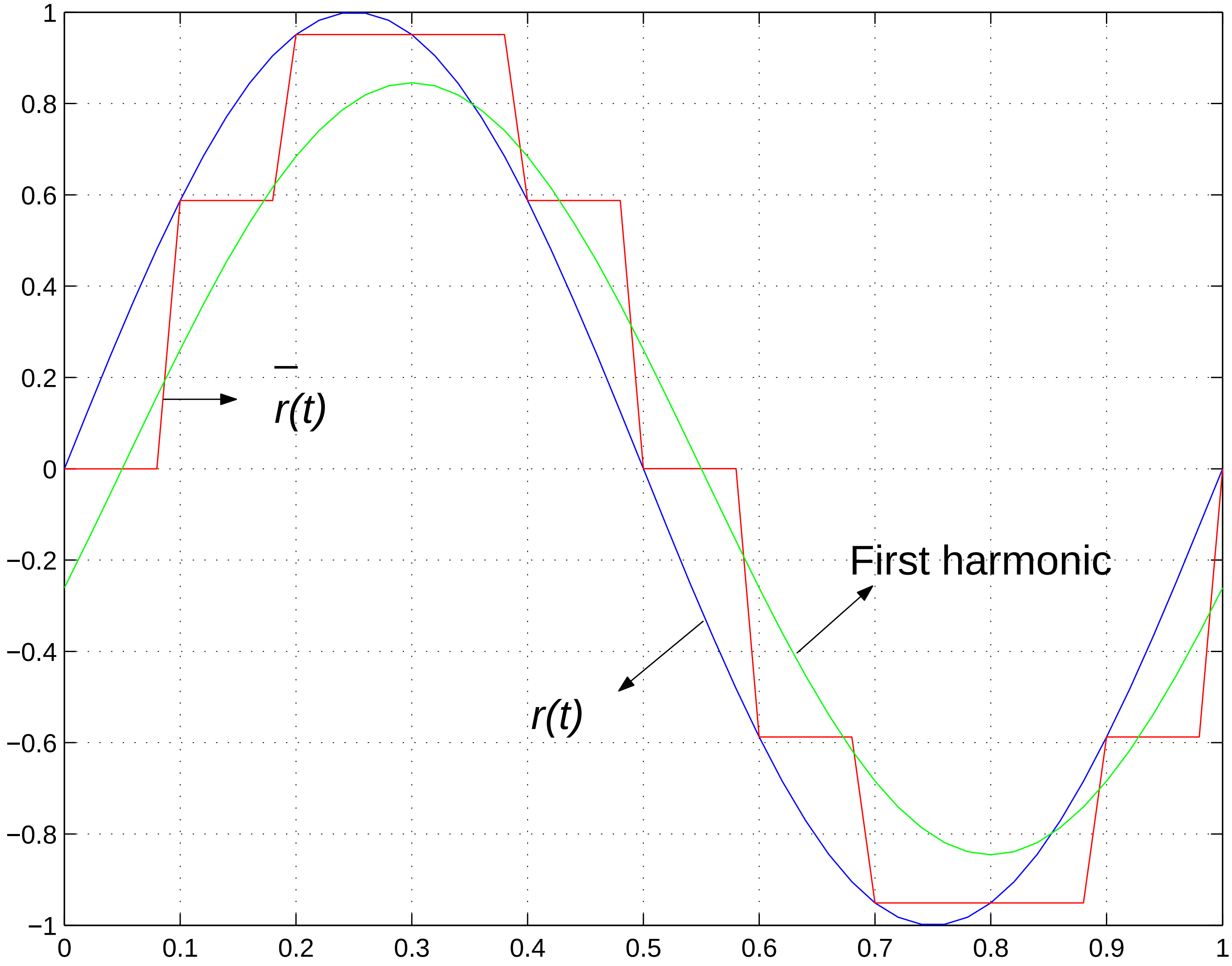
Harmonic Response of Sample and Hold: An Example



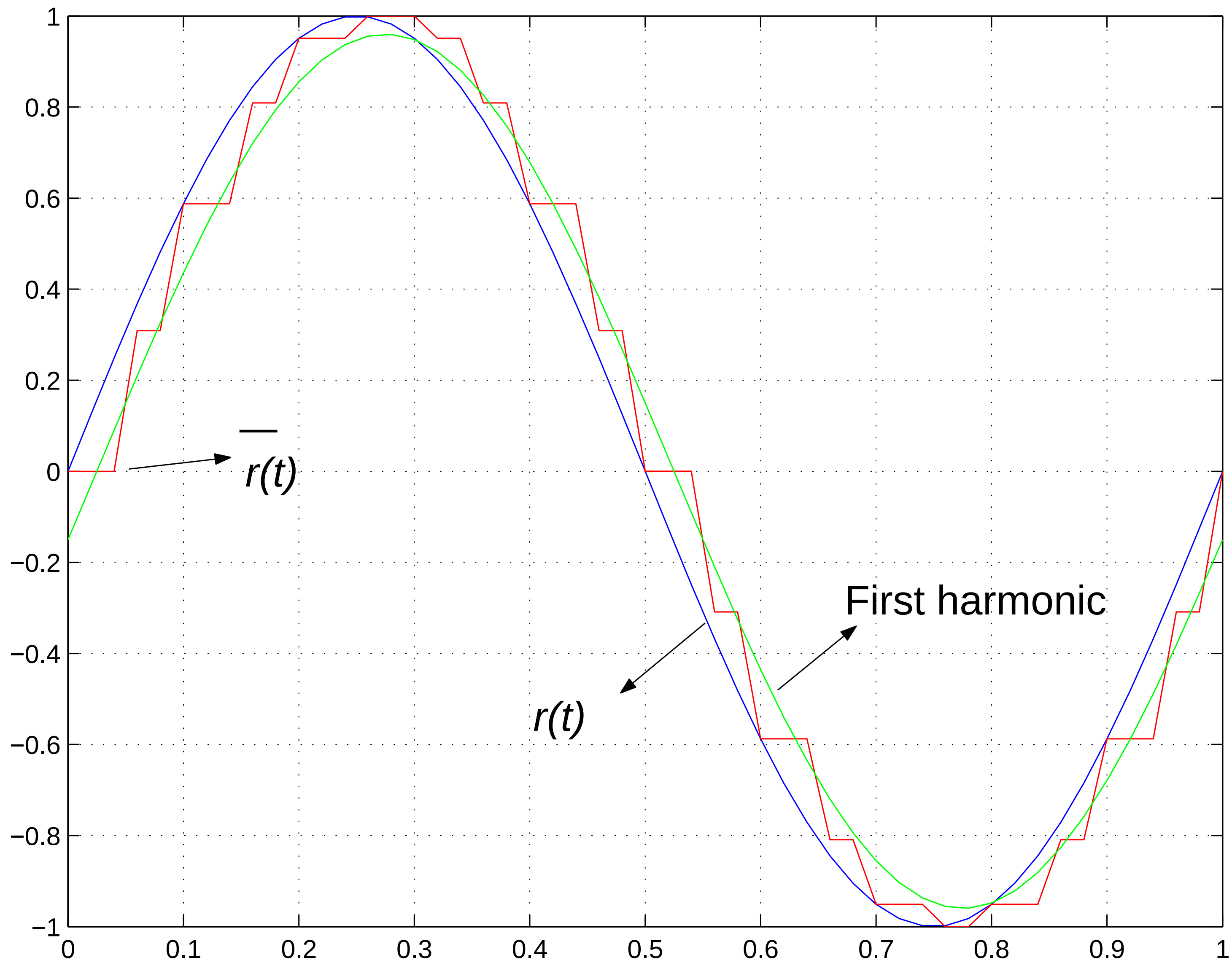
Harmonic Response of Sample and Hold: An Example



Harmonic Response of Sample and Hold: An Example



Harmonic Response of Sample and Hold: An Example



Higher-Order Hold Functions

- Interpolation using Taylor series

$$r(t) = r(nT) + \dot{r}(nT)(t - nT) + \tfrac{1}{2}\ddot{r}(nT)(t - nT)^2 + \cdots, \; nT \le t < (n+1)T$$

- Zero-order hold: Truncate at first term

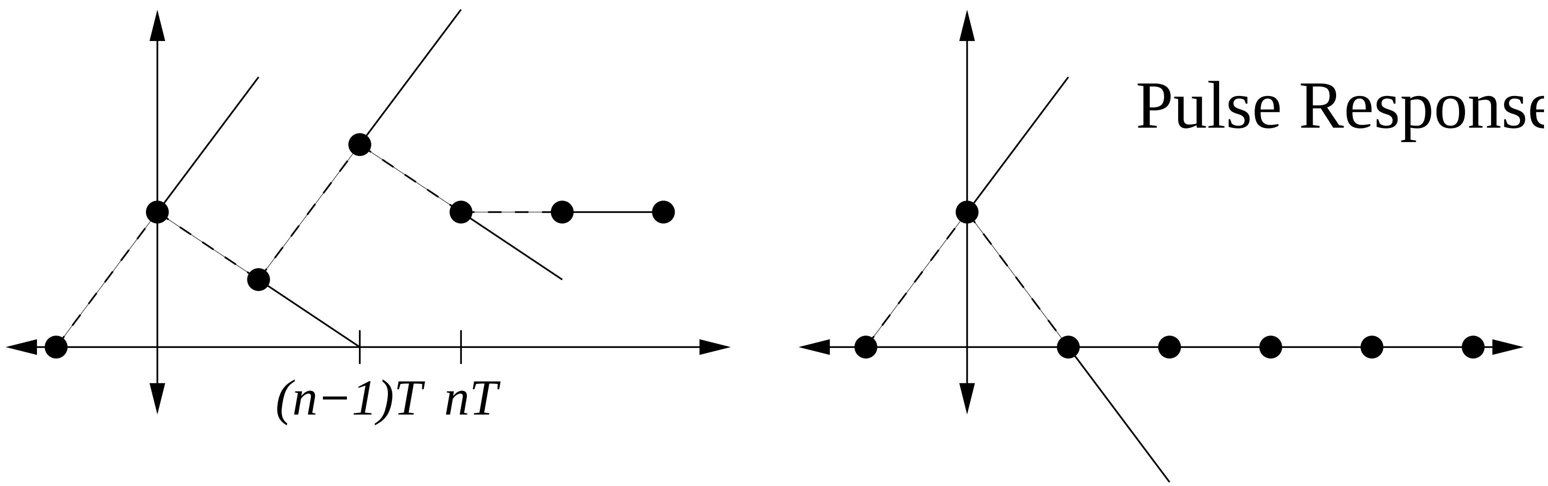
$$r(t) = r(nT), \; nT \le t \le (n+1)T$$

- First-order hold: Truncate at first-order term

$$r(t) = r(nT) + \dot{r}(nT)(t - nT), \; nT \le t \le (n+1)T$$

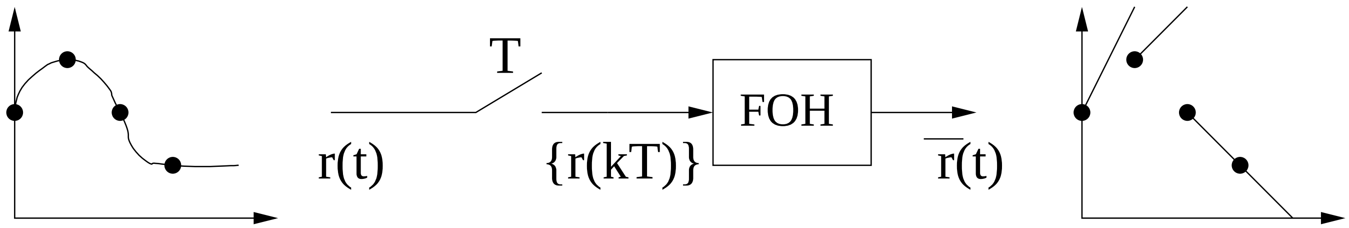
  – To find $\dot{r}(nT)$, extrapolate to $(n-1)T \le t \le (n+1)T$, put $t = (n-1)T$

$$\dot{r}(nT) = \frac{r(nT) - r((n-1)T)}{T}$$

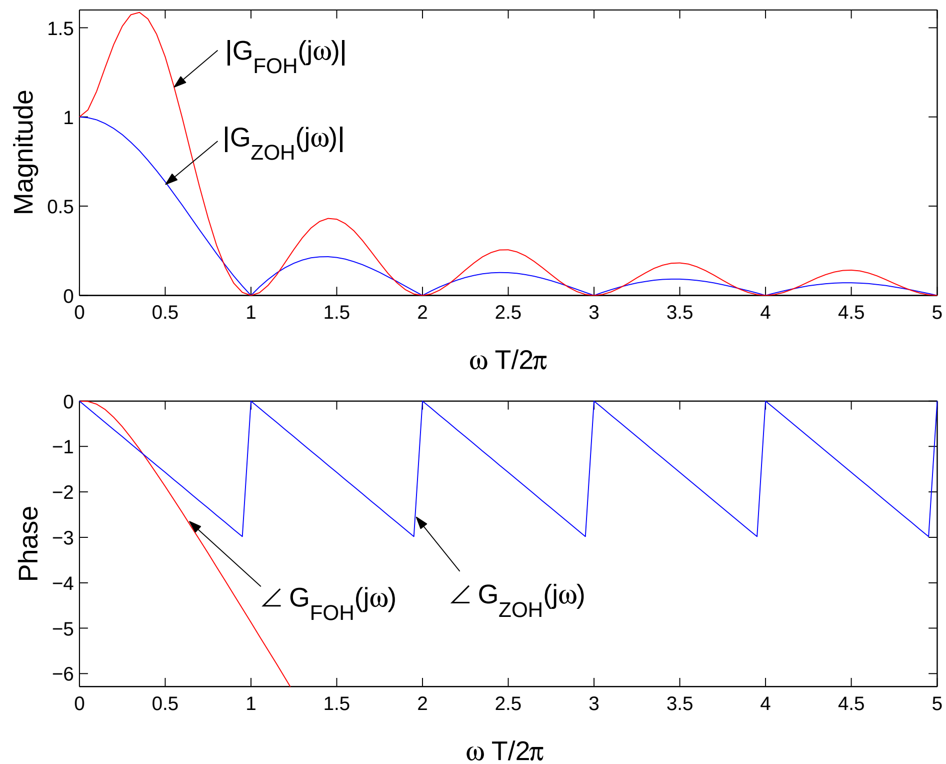


$$
\begin{aligned}
\text{Pulse Response} &= \bar{s}(t) + \frac{t}{T}\bar{s}(t) \\
&\quad -2\bar{s}(t-T) - \frac{2}{T}(t-T)\bar{s}(t-T) \\
&\quad +\bar{s}(t-2T) + \frac{1}{T}(t-2T)\bar{s}(t-2T). \\
G_{\mathrm{FOH}}(s) &= \frac{1}{s} - \frac{1}{s}2e^{-sT} + \frac{1}{s}e^{-2sT} + \frac{1}{Ts^2}(1 - 2e^{-sT} + e^{-2sT}) \\
&= \frac{(1+Ts)}{T}\left(\frac{1-e^{-sT}}{s}\right)^2
\end{aligned}
$$

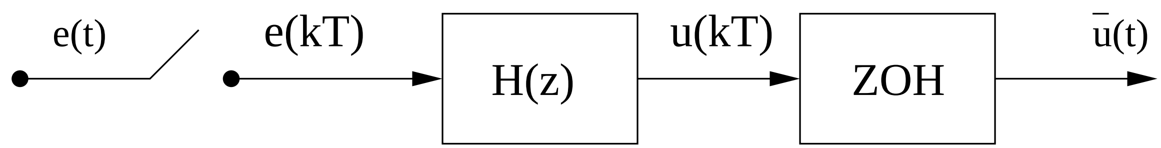
Analysis of a Sampler and First-Order Hold



$$\overline{R}(s) = R^*(s)G_{\text{FOH}}(s)$$

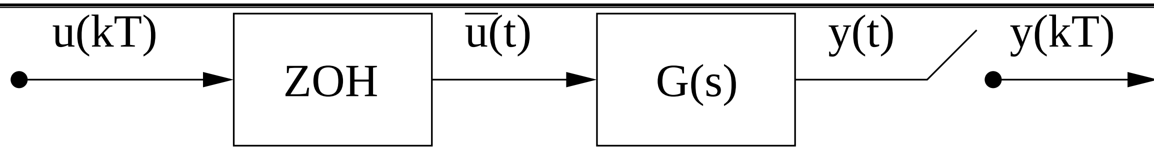


Analysis of a Sample, Process and Hold



$$
\begin{aligned}
\overline{U}(s) &= U^*(s)G_{\mathrm{ZOH}}(s) \\
&= U(z)|_{z=e^{sT}}\, G_{\mathrm{ZOH}}(s) \\
&= [H(z)E(z)]|_{z=e^{sT}}\, G_{\mathrm{ZOH}}(s) \\
&= \underbrace{H^*(s)E^*(s)}_{\text{convolved impulse trains}} \quad G_{\mathrm{ZOH}}(s)
\end{aligned}
$$

ZOH Equivalent



- Transfer Function possible
- Let $u(kT) = \delta(kT)$, unit pulse sequence

$$\begin{aligned} \bar{u}(t) &= \bar{s}(t) - \bar{s}(t-\tau) \\ y(t) &= w(t) - w(t-\tau),\ w = \text{unit step response of.} G(s) \\ y(kT) &= w(kT) - w((k-1)T) \\ Y(z) &= (1-z^{-1})W(z) \\ W(z) &= \mathcal{Z}[\mathcal{L}^{-1}(s^{-1}G(s))] = \mathcal{Z}(s^{-1}G(s)) \end{aligned}$$

$$\text{Transfer Function} = \underbrace{(1-z^{-1})\mathcal{Z}(s^{-1}G(s))}_{\text{ZOH equivalent}}$$

.

$$Y^*(s) = Y(z)|_{z=e^{sT}} = (1-e^{-sT})\mathcal{Z}(s^{-1}G(s))|_{z=e^{sT}}U^*(s)$$

$$Y(s) = G(s)\bar{U}(s) = G(s)G_{\text{ZOH}}(s)U^*(s)$$

A Glossary of Notation

$$G_{\text{ZOH}}(s) = \frac{1 - e^{-sT}}{s}$$

- Given $U(s)$

$$\begin{aligned} \mathcal{Z}(U(s)) &= \mathcal{Z} \text{ transform of sampled } u(t) \\ U^*(s) &= \mathcal{Z}(U(s))\big|_{z=e^{sT}} \end{aligned}$$

- Given $U(z)$

$$U^*(s) = U(z)\big|_{z=e^{sT}}$$

- Given $G(s)$

$$G_{\text{h}_0}(z) \stackrel{\text{def}}{=} \text{ZOH equivalent of } G(s) = (1 - z^{-1})\mathcal{Z}(s^{-1}G(s))$$

An Example

- No transfer function possible between $y(t)$ and $e(t)$
- Can find $Y(s)$
- Transfer function possible between $y(kT)$ and $e(kT)$

An Example

- No transfer function possible between $y(t)$ and $e(t)$
- Can find $Y(s)$
- Transfer function possible between $y(kT)$ and $e(kT)$

$$Y(s) = G(s)\overline{U}(s) = G(s)G_{\mathrm{ZOH}}(s)U^*(s) = G(s)G_{\mathrm{ZOH}}(s)H^*(s)E^*(s)$$

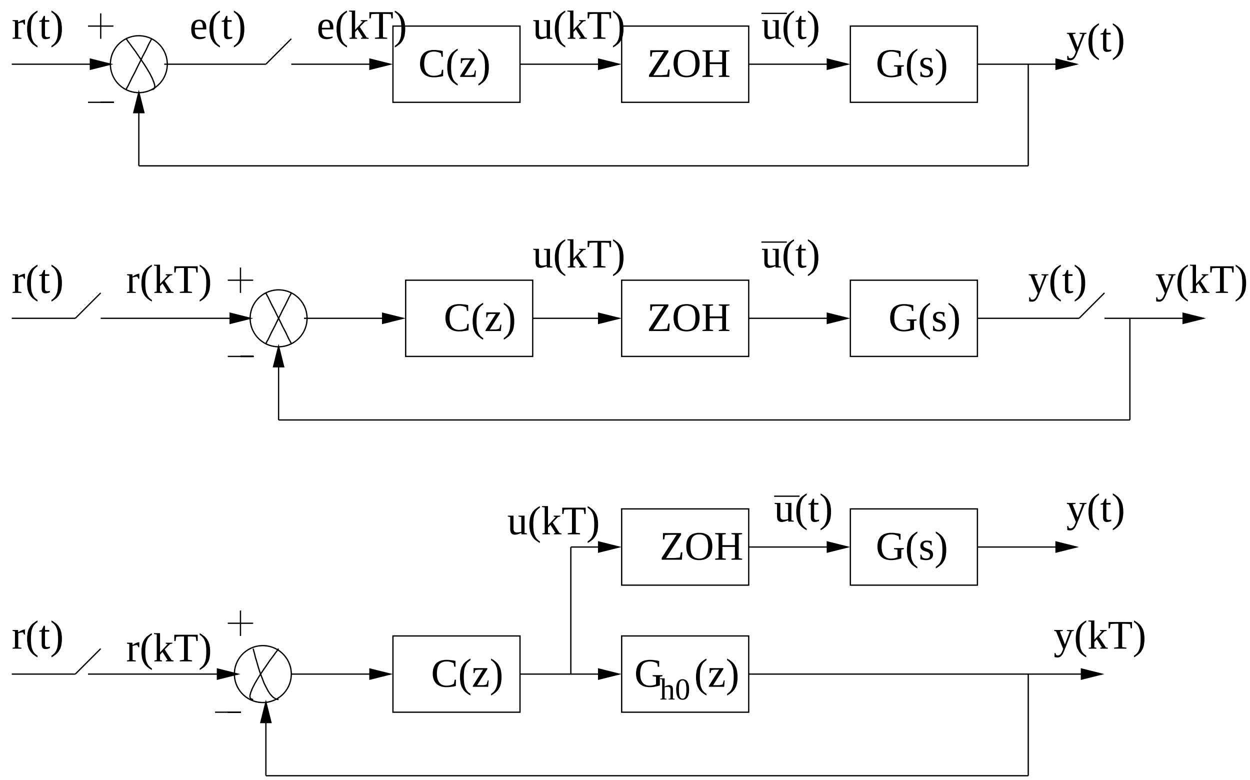
An Example

- No transfer function possible between $y(t)$ and $e(t)$
- Can find $Y(s)$
- Transfer function possible between $y(kT)$ and $e(kT)$

$$Y(s) = G(s)\overline{U}(s) = G(s)G_{\text{ZOH}}(s)U^*(s) = G(s)G_{\text{ZOH}}(s)H^*(s)E^*(s)$$

$$\frac{Y(z)}{E(z)} = H(z)G_{\text{h}_0}(z) = H(z)(1 - z^{-1})\mathcal{Z}(s^{-1}G(s))$$

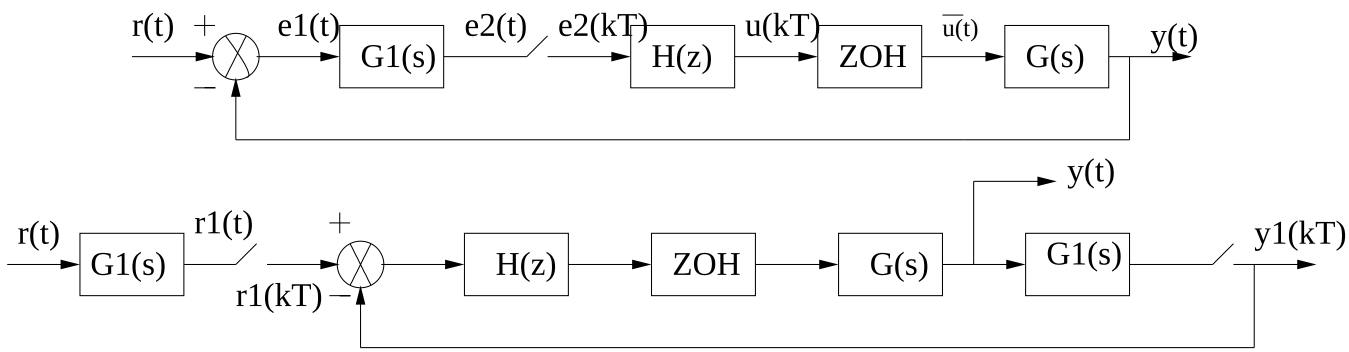
Block Diagram Manipulation For Sampled Data System: An Example



$$\frac{Y(z)}{E(z)} = \frac{C(z)G_{\mathrm{h_0}}(z)}{1 + C(z)G_{\mathrm{h_0}}(z)}$$

$$Y(s) = G(s)G_{\mathrm{ZOH}}(s)U^*(s) = G(s)G_{\mathrm{ZOH}}(s)C^*(s)(R^*(s) - Y^*(s))$$

Another Example



$$\frac{Y_1(z)}{R_1(z)} = \frac{H(z)(GG_1)_{h_0}(z)}{1 + H(z)(GG_1)_{h_0}(z)}, \quad Y_1^*(s) = \left(\frac{Y_1(z)}{R_1(z)}\right)\bigg|_{z=e^{sT}} R_1^*(s)$$

$$Y(s) = G(s)G_{\mathrm{ZOH}}(s)H^*(s)[R_1^*(s) - Y_1^*(s)]$$

$$R_1^*(s) = (GR_1)^*(s) \neq G^*(s)R^*(s)$$

Discrete-Time Equivalents of Continuous-Time Controllers

- Design controller in continuous time
- Numerically implement a discrete-time equivalent
- Example

$$H(s) = \frac{1}{s+a} \Longrightarrow \dot{y} + ay = u$$

$$\Longrightarrow y(t) = \int_0^t [u(\tau) - ay(\tau)]d\tau$$

$$\Longrightarrow y(kT) = y(kT - T) + \int_{(k-1)T}^{kT} [u(\tau) - ay(\tau)]d\tau$$

- Each numerical approximation for the integral gives a discrete-time equivalent

Backward Rectangular Rule

$$\int_{(k-1)T}^{kT} u(\tau)d\tau \approx u(kT)T$$

$$y(kT) = y(kT-T) + \int_{(k-1)T}^{kT} [u(\tau) - ay(\tau)]d\tau$$

$$y(kT) = y(kT-T) + Tu(kT) - aTy(kT)$$

$$\frac{Y(z)}{U(z)} = \frac{T}{1 - z^{-1} + aT} = \frac{1}{\left(\frac{1-z^{-1}}{T}\right) + a}$$

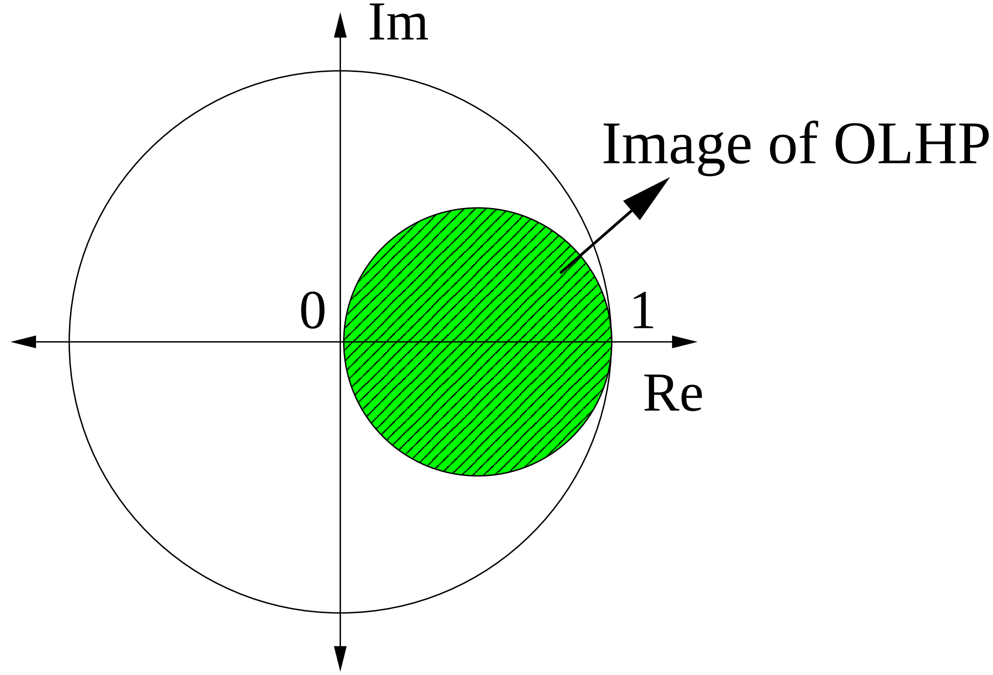
$$H_{\mathrm{B}}(z) = H(s)\big|_{s=(1-z^{-1})/T}$$

$$s \longleftrightarrow \frac{1-z^{-1}}{T}, \ z \longleftrightarrow \frac{1}{1-Ts}$$

Stability Regions Under Backward Rule

$$|z - \tfrac{1}{2}| = \frac{1}{2}\left|\frac{1+Ts}{1-Ts}\right|$$

$$\mathrm{Re}\ s < 0 \Longrightarrow |z - \tfrac{1}{2}| < \tfrac{1}{2}$$



Forward Rectangular Rule

$$\int_{(k-1)T}^{kT} u(\tau)d\tau \approx u(kT-T)T$$

$$y(kT) = y(kT-T) + \int_{(k-1)T}^{kT} [u(\tau) - ay(\tau)]d\tau$$

$$y(kT) = y(kT-T) + Tu(kT-T) - aTy(kT-T)$$

$$\frac{Y(z)}{U(z)} = \frac{T}{z-1+aT} = \frac{1}{\left(\frac{z-1}{T}\right)+a}$$

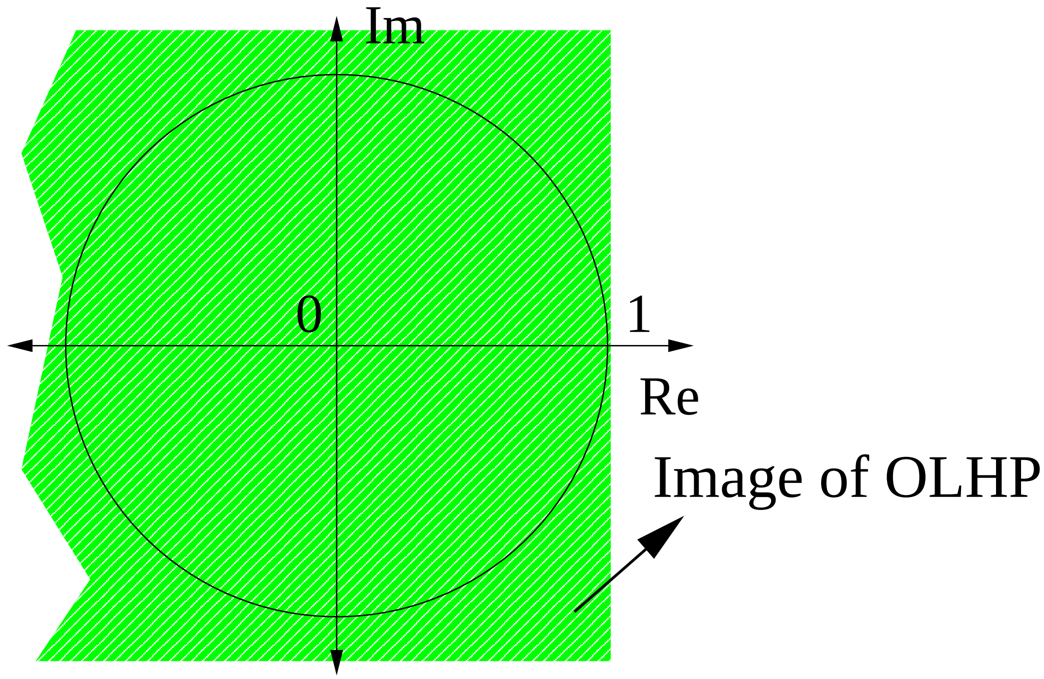
$$H_{\mathrm{F}}(z) = H(s)\big|_{s=(z-1)/T}$$

$$s \longleftrightarrow \frac{z-1}{T},\ z \longleftrightarrow 1+Ts$$

Stability Regions Under Forward Rule

$$z = 1 + Ts$$

$$\text{Re } s < 0 \Longrightarrow \text{Re } z < 1$$



Trapezoidal Rule

$$\int_{(k-1)T}^{kT} u(\tau)d\tau \approx \frac{T}{2}[u(kT-T)+u(kT)]$$

$$y(kT) = y(kT-T) + \int_{(k-1)T}^{kT} [u(\tau) - ay(\tau)]d\tau$$

$$y(kT) = y(kT-T) + \frac{T}{2}[u(kT-T)+u(kT)] - \frac{aT}{2}[y(kT-T)+y(kT)]$$

$$\frac{Y(z)}{U(z)} = \frac{1}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

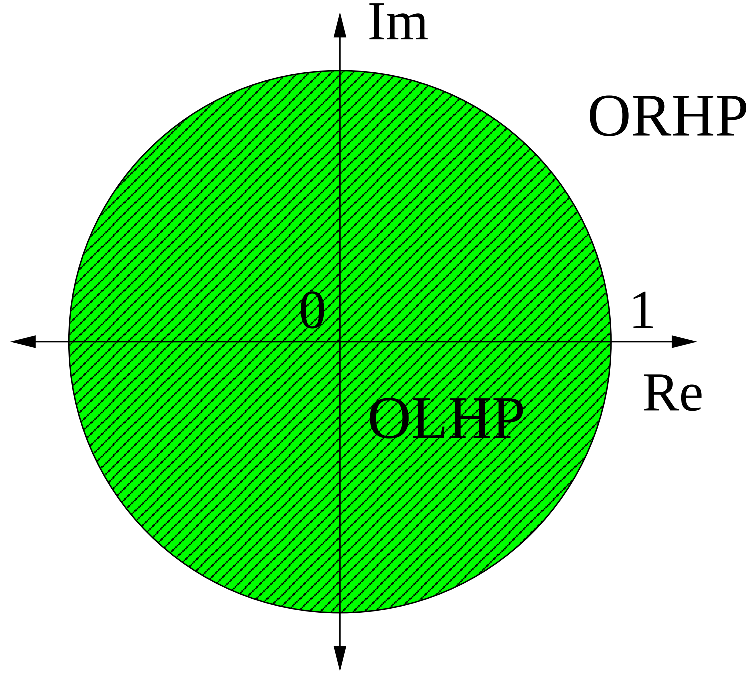
$$H_{\mathrm{T}}(z) = H(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} : \text{ Tustin's Rule}$$

$$s \longleftrightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}, \; z \longleftrightarrow \frac{1+Ts/2}{1-Ts/2}$$

Stability Regions Under Tustin's Rule

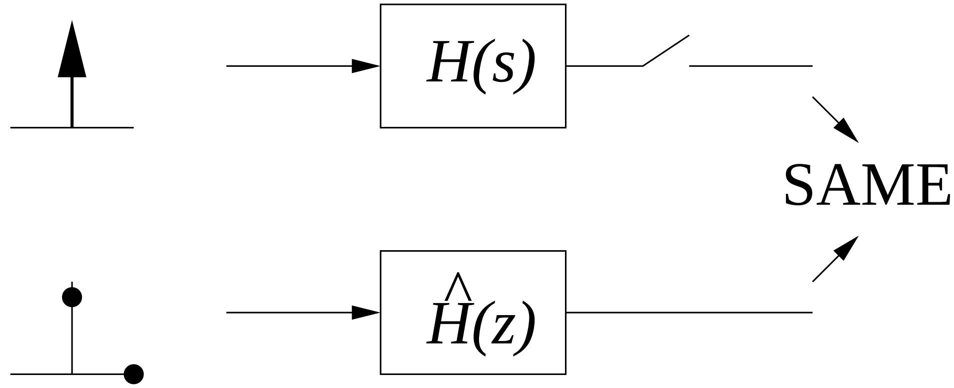
$$z = \frac{1 + Ts/2}{1 - Ts/2}$$

$$\text{Re } s < 0 \Longrightarrow |z| < 1$$



Discrete-Time Equivalent by Impulse Invariance

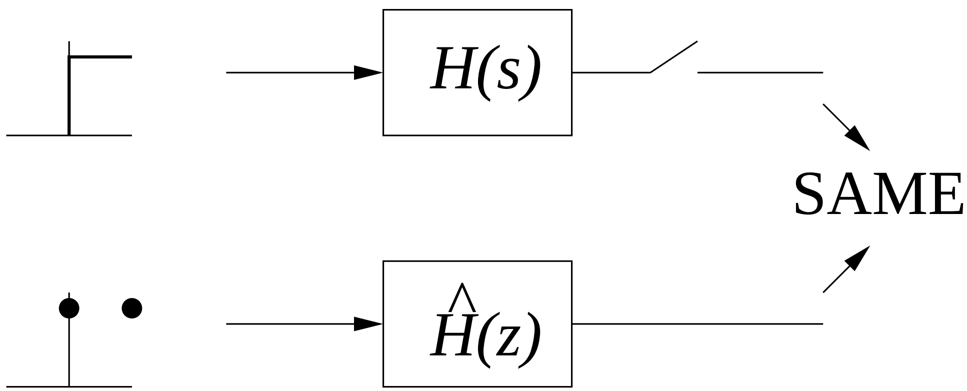
- Find $\widehat{H}(z)$ such that pulse response of $\widehat{H}(z)$ is the sampled sequence of the impulse response of $H(s)$



$$\widehat{H}(z) = \mathcal{Z}(H(s))$$

Discrete-Time Equivalence by Step Invariance

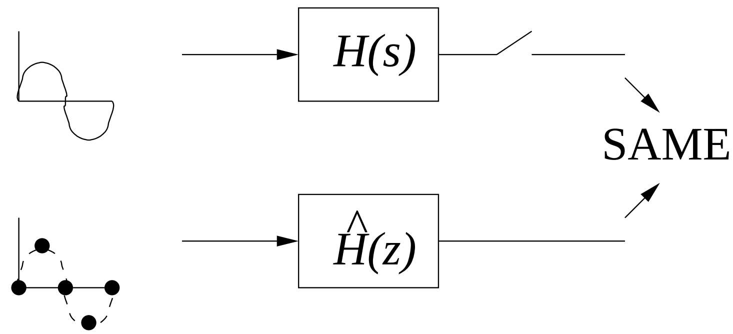
- Find $\widehat{H}(z)$ such that step response of $\widehat{H}(z)$ is the sampled sequence of the step response of $H(s)$



$$\widehat{H}(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{H(s)}{s}\right) = H_{h_0}(z)$$

Equivalence at a Frequency

- When will the steady state response of $\widehat{H}(z)$ to $\{\cos k\omega T\}$ equal the sampled sequence of the steady state response of $H(s)$ to $\cos \omega t$?



- If and only if

$$H(\jmath\omega) = \widehat{H}(e^{\jmath\omega T})$$

Tustin's Rule and Equivalence at a Frequency

$$H(s) = \frac{a}{s+a}, \;\; H_{\mathrm{T}}(z) = \frac{a}{\frac{2}{T}\frac{z-1}{z+1} + a}$$

$$H(\jmath a) = \frac{1}{1+\jmath}, \;\; H_{\mathrm{T}}(e^{\jmath aT}) = \frac{1}{1+\jmath\frac{2}{aT}\tan\frac{aT}{2}}$$

- The discrete equivalent does not "match" the original at the corner frequency
- Tustin's rule causes frequency distortion
- Distortion is reduced if $aT/2 << 1$

Tustin's Rule with Pre-warping

- Pre-warp the continuous system such that on applying Tustin's rule, matching is obtained at the selected frequency
- Substitute

$$s = b\frac{z-1}{z+1}$$

  - Recover Tustin's rule if $b = 2/T$
  - Same as applying Tustin's rule to the "pre-warped" transfer function

$$H_{\text{pre-warped}}(s) = H(bTs/2)$$

- Choose b to get matching at the desired frequency

Pole-Zero Mapping Equivalent

- Map all poles of $H(s)$ according to $z = e^{sT}$

$$\frac{1}{s+a} \mapsto \frac{1}{1 - e^{-aT} z^{-1}}$$

- Map all finite zeros of $H(s)$ by $z = e^{sT}$

$$(s+a) \mapsto 1 - e^{-aT} z^{-1}$$

- Map zeros at ∞ to zeros at -1

$$\frac{1}{s} \mapsto 1 + z^{-1}$$

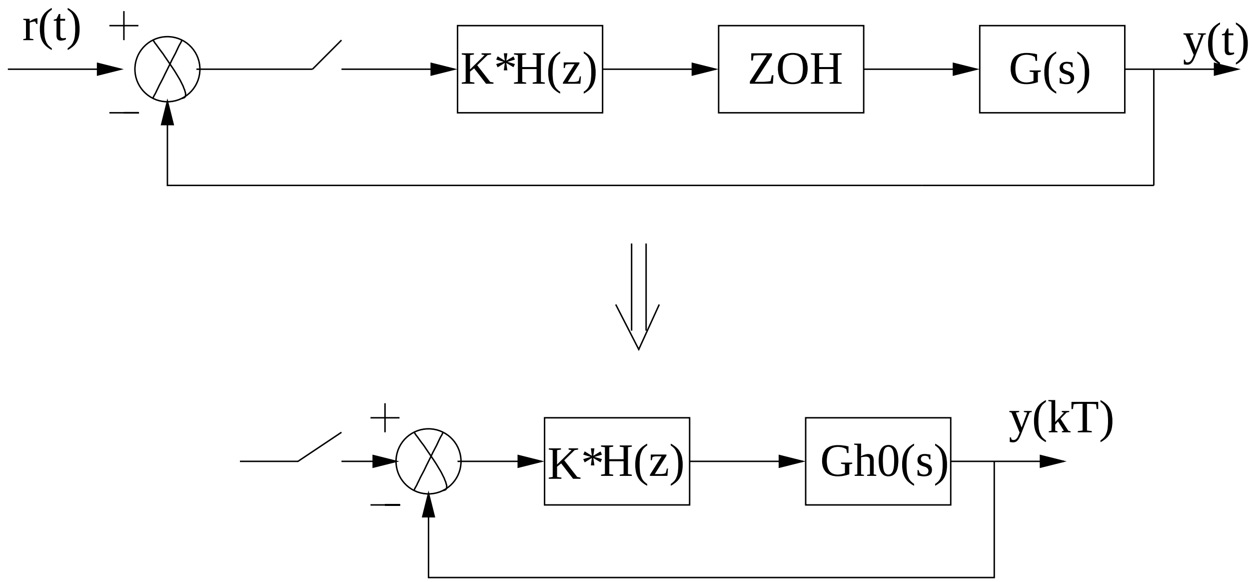
- To get a strictly causal system, map one s^{-1} factor to z^{-1}

- Choose gain factor to get matching at a specified frequency

$$H(\jmath\omega) = H_{\text{zp}}(e^{\jmath\omega T})$$

  – Usually $\omega = 0$, that is, matching at DC

Root Locus



- Root locus = locus of roots of $1 + KH(z)G_{h0}(z) = 0$ as K varies fro 0 to ∞
- Plotted in the same way as for continuous-time systems

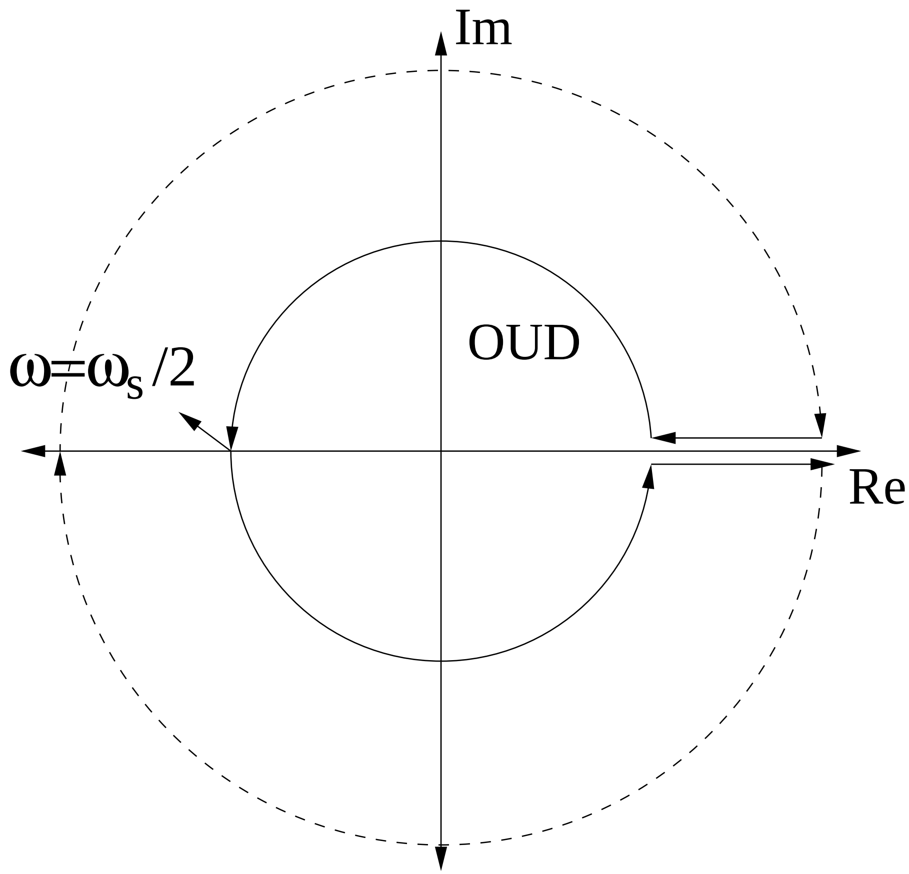
Mapping Theorem

- Based on Mapping Theorem
  - z traces a simple closed curve C clockwise in the complex plane
  - The no. clockwise of encirclements of the origin by $F(z)$ equals

    no. of zeros of F enclosed by C $-$ no of poles of F enclosed by C
- Application to closed-loop stability analysis
  - Choose $F(z) = 1 + G(z)H(z) =$ closed-loop characteristic polynomial
  - Choose C to enclose all possible unstable poles

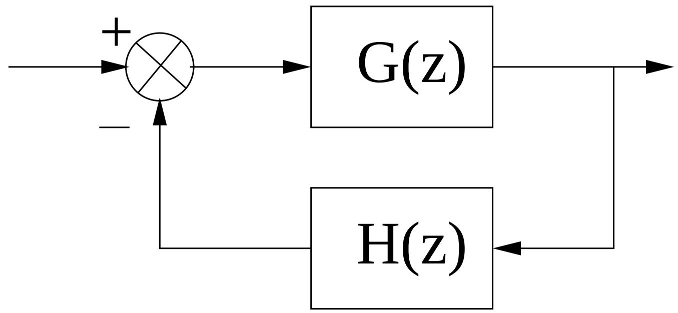
Nyquist Contour

- Choose C to enclose the exterior of the open unit disc



- All encirclements are contributed by portion along the unit circle

Nyquist Criterion



Loop transfer function $L(z) = G(z)H(z)$

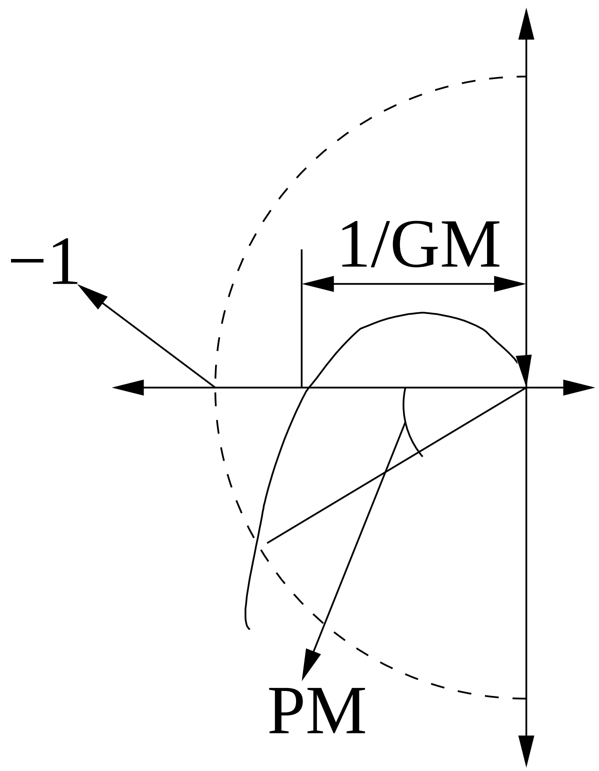
Nyquist Criterion:

$$Z = N + P$$

P = no. of unstable open-loop poles (unstable poles of $L(z)$)

Z = no. of closed-loop unstable poles (unstable roots of $1 + L(z) = 0$)

N = no. of clockwise encirclements of -1 by $L(e^{\jmath\omega T})$, $\omega \in [0, \omega_{\mathrm{s}}]$


−1
1/GM
PM

Frequency Response Analysis with $\mathcal{W}$-Transform

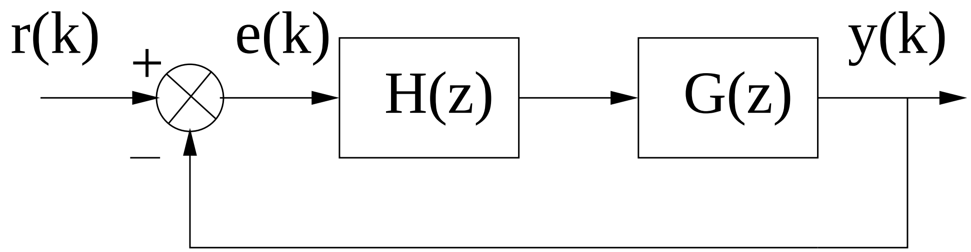
- Frequency response in terms of $\mathcal{Z}$-transform is
  - Periodic in ω
  - Difficult to draw by hand (s-domain rules do not apply)
- Use $\mathcal{W}$-transform to map OUD into OLHP using

$$w = \frac{2}{T}\frac{(z-1)}{(z+1)}, \; z = \frac{1+wT/2}{1-wT/2}$$

$$\widehat{G}(w) = G(z)\Big|_{z=\frac{1+wT/2}{1-wT/2}}$$

- Bode plots of $\widehat{G}(w)$ can be drawn using s-domain rules
- Nyquist criterion can be applied to $\widehat{G}(w)$ as in s-domain
- Controller $\widehat{H}$ designed for $\widehat{G}$ can be transformed back and applied to G
- $\widehat{G}$ and G yield the same gain and phase margins

Closed-Loop Asymptotic Tracking of Reference Inputs



$$\frac{E(z)}{R(z)} = \frac{1}{1 + G(z)H(z)}$$

- Asymptotic Tracking: Want

$$\lim_{k \to \infty} e(k) = 0$$

  - $\lim_{k \to \infty} e(k)$ exists if and only if all poles of $E(z)$ lie in the OUD except possibly for one pole at $z = 1$
  - $\lim_{k \to \infty} e(k)$, if it exists, equals $\lim_{z \to 1}(z - 1)E(z)$

Tracking of Step Inputs

$$r(k) = 1,\ R(z) = \frac{z}{z-1}$$

$$E(z) = \frac{z}{(z-1)} \frac{1}{[1 + G(z)H(z)]}$$

- For $\lim_{k\to\infty} e(k)$ to exist, all closed-loop poles must lie in the OUD

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1} \frac{z}{1 + G(z)H(z)} = \frac{1}{1 + \lim_{z\to 1} G(z)H(z)}$$

- For $\lim_{k\to\infty} e(k) = 0$, the (open) loop transfer function must have a pole at $z = 1$

- No. of poles of $G(z)H(z)$ at $z = 1$ is the *type* of the open-loop system

- Define position error constant

$$K_{\mathrm{p}} = \lim_{z\to 1} G(z)H(z)$$

- For perfect tracking, need $K_{\mathrm{p}} = \infty$

For perfectly tracking step inputs, need closed-loop stability + type 1 open-loop system

Tracking of Ramp Inputs

$$r(k) = kT, \;\; R(z) = \frac{Tz}{(z-1)^2}$$

$$E(z) = \frac{Tz}{(z-1)^2}\frac{1}{[1+G(z)H(z)]} = \frac{Tz}{(z-1)[z-1+(z-1)G(z)H(z)]}$$

- For $\lim_{k\to\infty} e(k)$ to exist, all closed-loop poles must lie in the OUD, and open-loop system must be of type 1

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1}\frac{Tz}{(z-1)[1+G(z)H(z)]} = \frac{T}{\lim_{z\to 1}(z-1)G(z)H(z)}$$

- For $\lim_{k\to\infty} e(k) = 0$, the (open) loop transfer function must have at least two poles at $z = 1$

- Define velocity error constant

$$K_\mathrm{v} = \lim_{z\to 1}(z-1)G(z)H(z)/T$$

- For perfect tracking, need $K_\mathrm{v} = \infty$

For perfectly tracking ramp inputs, need closed-loop stability + type 2 open-loop system

Tracking of Sinusoidal Inputs

$$r(k) = A\sin(k\omega T),\ \ R(z) \text{ has poles at } e^{\pm \jmath\omega T}$$

- For $e(k)$ to converge to a steady state behavior, closed-loop must be (BIBO) stable
- Steady-state error amplitude

$$= \frac{1}{|1 + G(e^{\jmath\omega T})H(e^{\jmath\omega T})|}$$

- For $\lim_{k\to\infty} e(k) = 0$, $G(z)H(z)$ must have at least one pole at $z = e^{\jmath\omega T}$

For perfectly tracking $\{A\sin k\omega T\}$, need closed-loop stability + open-loop poles at $z = e^{\pm\jmath\omega T}$

Internal Model Principle

- Requirements for closed-loop tracking

Input to be tracked		Requirements for tracking	
Input	Input poles	Open-loop poles	Closed-loop poles
Step	$z = 1$	$z = 1$	OUD
Ramp	$z = 1, 1$	$z = 1, 1$	OUD
Sinusoidal	$z = e^{\pm \jmath \omega T}$	$z = e^{\pm \jmath \omega T}$	OUD

- Internal Model Principle: A closed-loop system will track an input perfectly asymptotically if and only if
  - The closed-loop system is stable and
  - The open-loop system contains a "model" of the input

Continuous-Time LTI State-Space Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, && \text{state dynamics} \\ y &= Cx + Du, && \text{measurement equation} \end{aligned}$$

$$\begin{aligned} x &= \text{vector of state/internal variables} \\ y &= \text{vector of output measurements} \\ u &= \text{vector of inputs} \end{aligned}$$

- Linear: A, B, C, D independent of x, y
- Time Invariant: A, B, C, D independent of time
- To find output, need
  - Initial state vector and input

Matrix Exponential

- Given a square matrix A, define

$$e^{At} \stackrel{\text{def}}{=} I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

- Well defined (series converges sufficiently nicely)

- Satisfies

$$\begin{aligned} e^{A0} &= I \\ \frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A \\ e^{A(t+\tau)} &= e^{At}e^{A\tau} \\ (e^{At})^{-1} &= e^{-At} \end{aligned}$$

- Note: $(e^{At})_{ij} \neq e^{A_{ij}t}$

Examples

- $\ddot{y} + \omega_\mathrm{n}^2 y = \frac{1}{m}u$

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_\mathrm{n}^2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{k} \\ 0 \end{bmatrix}}_{B} u$$

$$e^{At} = \begin{bmatrix} \cos\omega_\mathrm{n}t & \frac{1}{\omega_\mathrm{n}}\sin\omega_\mathrm{n}t \\ -\omega_\mathrm{n}\sin\omega_\mathrm{n}t & \cos\omega_\mathrm{n}t \end{bmatrix}$$

- $\ddot{y} = u$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Solution of the State Equation

- State response

$$x(t) = \underbrace{e^{At}x(0)}_{\text{Natural Response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{Forced Response}}$$

$$e^{At} = \text{state transition matrix}$$

- Output response

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{Natural Response}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{\text{Forced Response}}$$

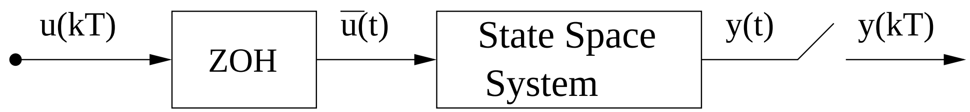
- Impulse response

$$g(t) = Ce^{At}B + D\delta(t)$$

- Transfer matrix

$$C(sI - A)^{-1}B + D$$

ZOH Equivalent of a State Space System



$$\overline{u}(t) = u(k),\ t \in [kT, kT+T)$$

$$\begin{aligned} x(k+1) &= e^{AT}x(k) + \int_{kT}^{kT+T} e^{A(kT+T-\tau)}Bu(k)d\tau \\ &= e^{AT}x(k) + \left[\int_0^T e^{A(T-\sigma)}Bd\sigma\right]u(k),\ \sigma = \tau - kT \end{aligned}$$

- Discrete-time state space model of the ZOH equivalent

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Hx(k) + Ju(k) \end{aligned}$$

$$\Phi = e^{AT},\ \Gamma = \left[\int_0^T e^{A(T-\sigma)}Bd\sigma\right],\ H = C, J = D$$

State Space Realizations

$$G(z) = \frac{b_0 + b_1 z^{-1} \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

- Let $\widehat{Y}(z) = U(z)[1 + a_1 z^{-1} + \cdots + a_n z^{-n}]^{-1}$, so that $Y(z) = [b_0 + b_1 z^{-1} \cdots + b_n z^{-n}]\widehat{Y}(z)$

$$\hat{y}(k) + a_1 \hat{y}(k-1) + \cdots + a_n \hat{y}(k-n) = u(k)$$

- Choose

$$x(k) = \begin{bmatrix} \hat{y}(k-n) \\ \hat{y}(k-n+1) \\ \vdots \\ \hat{y}(k-1) \end{bmatrix}, \; x(k{+}1) = \begin{bmatrix} \hat{y}(k-n+1) \\ \hat{y}(k-n+2) \\ \vdots \\ \hat{y}(k) \end{bmatrix} = \begin{bmatrix} x_2(k) \\ x_3(k) \\ \vdots \\ -a_n x_1(k) - \cdots - a_1 x_n(k) + u(k) \end{bmatrix}$$

$$\begin{aligned} y(k) &= b_0 \hat{y}(k) + b_1 \hat{y}(k-1) + \cdots + b_n \hat{y}(k-n) \\ &= b_0 \hat{y}(k) + b_1 x_n(k) + \cdots + b_n x_1(k) \\ &= (b_n - b_0 a_n) x_1(k) + (b_{n-1} - b_0 a_{n-1}) x_2(k) + \cdots + (b_1 - b_0 a_1) x_n(k) + b_0 u(k) \end{aligned}$$

State Space Realizations (cont'd)

- First companion form

$$
\begin{aligned}
x(k+1) &= \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B} u(k) \\
y(k) &= \underbrace{\begin{bmatrix} (b_n - b_0 a_n) & \cdots & (b_1 - b_0 a_1) \end{bmatrix}}_{C} x(k) + \underbrace{[b_0]}_{D} u(k)
\end{aligned}
$$

Transfer Matrix from State Space Model

- State dynamics equations in Laplace domain

$$zX(z) - zx(0) = AX(z) + BU(z)$$

$$X(z) = \underbrace{z(zI - A)^{-1}x(0)}_{\text{Natural response}} + \underbrace{(zI - A)^{-1}BU(z)}_{\text{Forced response}}$$

$$\begin{aligned} Y(z) &= CX(z) + DU(z) \\ &= Cz(zI - A)^{-1}x(0) + \underbrace{\left[C(zI - A)^{-1}B + D\right]}_{\text{Transfer matrix}} U(s) \end{aligned}$$

$$\text{Transfer matrix } = C(zI - A)^{-1}B + D$$

- Poles are eigenvalues of A

Transformations of State Space Models

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

- State transformation $\hat{x} = Sx$

$$\begin{aligned} \hat{x}(k+1) &= \underbrace{SAS^{-1}}_{\widehat{A}}\hat{x}(k) + \underbrace{SB}_{\widehat{B}}u(k) \\ y(k) &= \underbrace{CS^{-1}}_{\widehat{C}}\hat{x}(k) + \underbrace{D}_{\widehat{D}}u(k) \end{aligned}$$

- Input-output relation is unchanged

$$C(zI - A)^{-1}B + D = \widehat{C}(zI - \widehat{A})^{-1}\widehat{B} + \widehat{D}$$

- Two state-space models are *equivalent* if they yield the same transfer matrix
- Every input-output system has several equivalent state space representations/realizations

State Evolution in Discrete-Time System

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
x(1) &= Ax(0) + Bu(0) \\
x(2) &= A^2x(0) + ABu(0) + Bu(1) \\
x(3) &= A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2) \\
&\vdots \\
x(k) &= \underbrace{A^k x(0)}_{\text{natural response}} + \underbrace{\sum_{l=0}^{k-1} A^{k-l-1}Bu(l)}_{\text{forced response}} \\
y(k) &= \underbrace{CA^k x(0)}_{\text{natural response}} + \underbrace{\sum_{l=0}^{k-1} CA^{k-l-1}Bu(l) + Du(k)}_{\text{forced response}}
\end{aligned}
$$

Impulse Response

- Response to impulse input $u(k) = u_0\delta(k)$

$$y(k) = CA^{k-1}Bu_0 + Du_0\delta(k)$$

- Impulse response sequence

$$= \{Du_0, CBu_0, CABu_0, \ldots\}$$

- Impulse response matrix

$$\begin{aligned} H(k) &= D,\ k = 0 \\ &= CA^{k-1}B,\ k > 0 \end{aligned}$$

- General response

$$y(k) = CA^k x(0) + (H * u)(k)$$

- Compare with forced response in z-domain

$$Y(z) = [C(zI - A)^{-1}B + D]U(z)$$

$$\Longrightarrow \mathcal{Z}(H) = C(zI - A)^{-1}B + D = D + z^{-1}CB + z^{-2}CAB + \cdots$$

Reachable Sets

- Which states can be reached from a given initial condition by using all possible inputs?
- Reachable set from x_0 at step k

$$\begin{aligned}\mathcal{R}(k, x_0) &= \{\text{states reachable from } x_0 \text{ in } k \text{ steps}\} \\ &= \left\{A^k x_0 + \sum_{l=0}^{k-1} A^{k-l-1} B u(l) : u(0), u(1), \ldots, u(k-1) \text{ arbitrary}\right\}\end{aligned}$$

- System is *controllable* if every state can be reached from every other state in a finite (but possibly large) number of steps
- System is controllable iff $\mathcal{R}(k, x_0) = \mathbb{R}^n$ for sufficiently large k

Facts on Reachable Sets

$$\text{Fact 1}: \mathcal{R}(k, x_0) = \mathcal{R}(k, 0) + A^k x_0$$

$$\text{Fact 2}: \mathcal{R}(n, 0) = \text{Range}\ \mathcal{C},\ \ \mathcal{C} = [B\ AB\ A^2B\ \cdots A^{n-1}B]$$

- If $a \in \mathcal{R}(n, 0)$, then

$$a = Bu(n-1) + ABu(n-2) + \cdots + A^{n-1}Bu(0) = \mathcal{C}\begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix} \in \text{Range}\ \mathcal{C}$$

- If $a \in \text{Range}\ \mathcal{C}$, then

$$a = \mathcal{C}b = Bb_1 + ABb_2 + \cdots + A^{n-1}Bb_n \in \mathcal{R}(n, 0)$$

Facts on Reachable Sets (cont'd)

Fact 3 : $\mathcal{R}(k,0) = \mathcal{R}(n,0), \; k \geq n$

- Clearly $\mathcal{R}(n,0) \subseteq \mathcal{R}(k,0)$
- For $k > n$, an element of $\mathcal{R}(k,0)$ is of the form

$$Bu(k-1) + \cdots + A^{n-1}Bu(k-n) + A^{n}Bu(k-n+1) + \cdots + A^{k-1}Bu(0)$$

- By Cayley-Hamilton theorem, powers of A higher than $n-1$ can be written as combinations of powers of A upto $n-1$
- Elements of $\mathcal{R}(k,0)$ are contained in $\mathcal{R}(n,0)$

System is controllable iff $\text{rank}\, \mathcal{C} = n$

Unobservable Sets

- Can we guess the initial state by observing only the output?

$$
\begin{aligned}
y_1(k) &= CA^k x_1 + (H * u)(k) \\
y_2(k) &= CA^k x_2 + (H * u)(k)
\end{aligned}
$$

- Can distinguish x_1 from x_2 iff $CA^k x_1 \neq CA^k x_2$ for some k
- Unobservable set from x_0 at step k

$$
\begin{aligned}
\mathcal{U}(k, x_0) &= \{\text{states that yield the same output as } x_0 \text{ upto step } k-1\} \\
&= \{x : CA^i x = CA^i x_0,\ i = 0, 1, \cdots, k-1\}
\end{aligned}
$$

- System is *observable* if every state can be distinguished from every other state in a finite (but possibly large) number of steps
- System is observable iff $\mathcal{U}(k, x_0) = \{x_0\}$ for sufficiently large k

Facts on Unobservable Sets

Fact 1 : $\mathcal{U}(k, x_0) = \mathcal{U}(k, 0) + x_0$

Fact 2 : $\mathcal{U}(k, 0) = \mathcal{U}(n, 0), \;\; k \geq n$

Fact 3 : $\mathcal{U}(n, 0) = \text{kernel } \mathcal{O}, \;\; \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

System is observable iff $\text{rank } \mathcal{O} = n$

Hautus Test for Controllability

- Eigenvalue $\lambda \in \mathbb{C}$ of A is *controllable* if

$$\text{rank } [\lambda I - A \ B] = n$$

- Fact: System is controllable iff every eigenvalue of A is controllable
- If λ is not controllable, then there exists $x \in \mathbb{C}^n$ such that

$$x^* A = \lambda x^*, x^* B = 0 \Longrightarrow x^* A^i B = 0 \Longrightarrow x^* \mathcal{C} = 0 \Longrightarrow \text{rank } \mathcal{C} < n$$

- Controllability remains invariant under state transformation
- Uncontrollable eigenvalues are unaffected by control
  - If λ is an uncontrollable eigenvalue, and the feedback $u = Kx$ is used, then λ also appears as a closed-loop eigenvalue

Hautus Test for Observability

- Eigenvalue $\lambda \in \mathbb{C}$ of A is *unobservable* if

$$\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$$

- Fact: System is observable iff every eigenvalue of A is observable
- If λ is not observable, then there exists $x \in \mathbb{C}^n$ such that

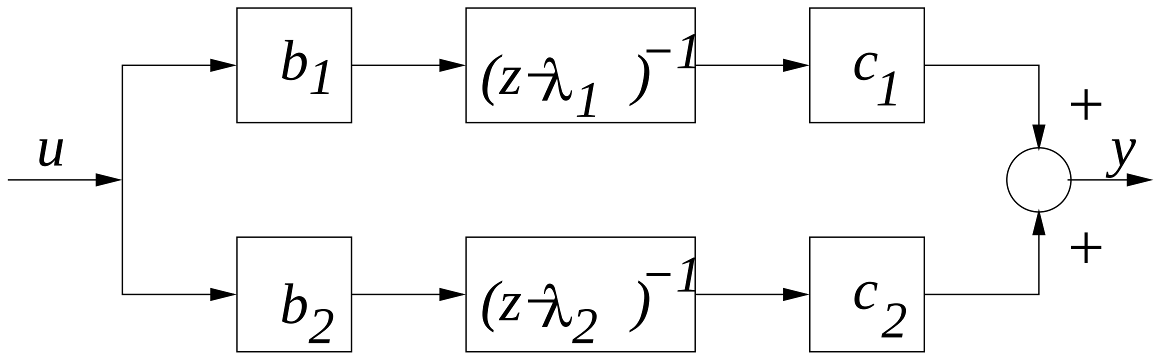
$$Ax = \lambda x, Cx = 0 \Longrightarrow CA^i x = 0 \Longrightarrow \mathcal{O}x = 0 \Longrightarrow \operatorname{rank} \mathcal{O} < n$$

- Observability remains unchanged under state transformations
- Unobservable eigenvalues cannot be detected through the output

A Two-Dimensional Example

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \; B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$$



- By the Hautus test, need $b_1 \neq 0 \neq b_2$ for controllability and $c_1 \neq 0 \neq c_2$ for observability

Kalman Decomposition Theorem

- Fact: Every state space system can be transformed into

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} A_{c\bar{o}} & A_{12} & A_{13} \\ 0 & A_{co} & A_{23} \\ 0 & 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_{c\bar{o}}(k) \\ x_{co}(k) \\ x_{\bar{c}}(k) \end{bmatrix} + \begin{bmatrix} B_{c\bar{o}} \\ B_{co} \\ 0 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 0 & C_{co} & C_{\bar{c}} \end{bmatrix} x(k) + Du(k)
\end{aligned}
$$

$$
G(s) = C(zI - A)^1 B + D = C_{co}(zI - A_{co})^{-1} B_{co} + D
$$

- The controllable and observable part yields a smaller realization
- Eigenvalues that are either unobservable or uncontrollable are not poles

Minimal Realizations

- A *minimal* realization is one having the least no. of states
  - Desirable for implementation
- A minimal realization has as many states as the number of poles
- A realization is minimal iff it is controllable and observable
- All minimal realizations are equivalent

Jordan Form

- Every matrix can be reduced to its *Jordan* form through a similarity transformation

$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

$$\text{characteristic polynomial} = (z-\lambda_1)^3(z-\lambda_2)^3(z-\lambda_3)^2(z-\lambda_4)$$

- λ_1 and λ_4 have 1 eigenvector each, λ_2 and λ_3 have 2 eigenvectors each

- λ_3 and λ_4 are *semisimple*, λ_4 is *simple*

Internal Stability

- Internal stability refers to the natural response of state (internal) variables
- A state space system is (internally)
  - *Lyapunov stable* if every initial condition response is bounded
  - *Asymptotically stable* if every initial condition response decays to zero
  - *Unstable* if it is not Lyapunov stable

$$x(k) = A^k x(0) = T J^k T^{-1} x(0), \; J = \text{ Jordan form}$$

- Stability depends on the elements of J^k

Powers of Jordan Blocks

$$J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \Longrightarrow J^k = \begin{bmatrix} \lambda^k & 0 \\ 0 & \lambda^k \end{bmatrix}$$

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Longrightarrow J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$$

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \Longrightarrow J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

- System is Lyapunov stable iff
  - All eigenvalues $\in$ CUD and
  - All eigenvalues of unit magnitude are semisimple
- System is asymptotically stable iff all eigenvalues $\in$ OUD

BIBO Stability and Internal Stability

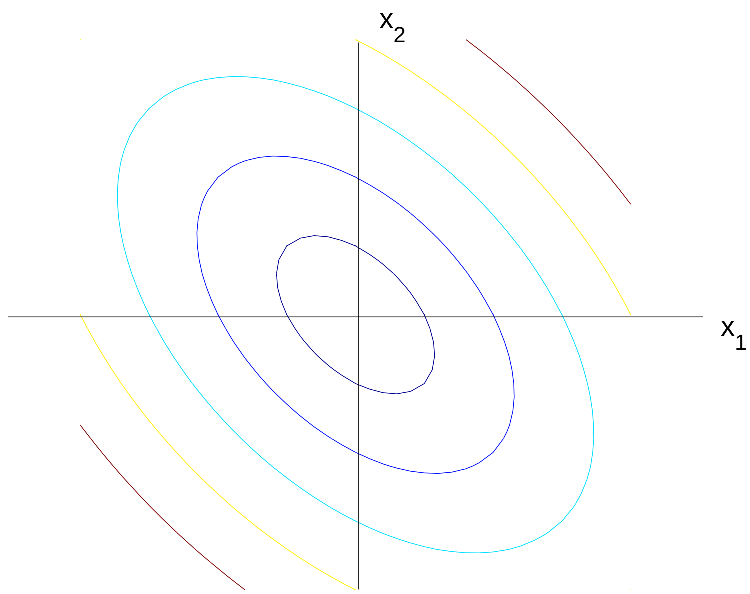
- System is BIBO stable iff every input vector with bounded components gives an output vector with bounded components
- System is BIBO stable if and only if every pole $\in$ OUD

$$\text{(internal) asymptotic stability} \Longrightarrow \text{BIBO stability}$$

- Converse does not hold in general
- Fact: A controllable, observable, BIBO stable system is asymptotically stable
- Fact: A system is BIBO stable iff every minimal realization is asymptotically stable

Positive-Definite Matrices

- $P \in \mathbb{R}^{n \times n}$, symmetric, is *positive-definite* $(P > 0)$ if $x^{\mathrm{T}} P x > 0$ for every $x \in \mathbb{C}^n$, $x \neq 0$
- A symmetric positive-definite matrix has real eigenvalues that are positive
- Every symmetric positive-definite matrix gives rise to the quadratic function $V_P(x) = x^{\mathrm{T}} P x$
- If $P > 0$, then the level sets of V_P are hyper-ellipsoids, eg. $P = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$



Level curves of V_P

Lyapunov Function

- How does a given quadratic function change along the natural state response?

$$
\begin{aligned}
x(k+1) &= Ax(k) \\
V_P(x(k)) &= x^{\mathrm{T}}(k)Px(k) \\
V_P(x(k+1)) &= x^{\mathrm{T}}(k+1)Px(k+1) = x^{\mathrm{T}}(k)A^{\mathrm{T}}PAx(k) \\
V_P(x(k+1)) - V_P(x(k)) &= x^{\mathrm{T}}(k)[A^{\mathrm{T}}PA - P]x(k)
\end{aligned}
$$

- Idea: If P is positive definite and $V_P(x(k))$ decreases with k, then $x(k) \longrightarrow 0$
  - Such a V_P is called a Lyapunov function
  - Want $P > 0$ and $A^{\mathrm{T}}PA - P = -Q$, where $Q > 0$

Lyapunov Equation

- Fact: If there exist $P > 0$ and $Q > 0$ satisfying the Lyapunov equation below, then system is asymptotically stable

$$A^{\mathrm{T}}PA - P = -Q$$

- Fact: System is asymptotically stable iff for every $Q > 0$, there exists a positive-definite solution P to the Lyapunov equation
  - For an asymptotically stable system, the solution P is unique
- To prove stability or instability, pick $Q > 0$ (eg. $Q = I$), solve for P and check sign definiteness of P
- OR check the feasibility of the linear matrix inequalities (LMIs)

$$\begin{aligned} -A^{\mathrm{T}}PA + P &> 0 \\ P &> 0 \end{aligned}$$

  - Can be done using efficient numerical algorithms

Full-State Feedback

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) && \text{Open – loop system} \\ u(k) &= -Kx(k) + r(k) && \text{Full – state feedback} \\ x(k+1) &= (A - BK)x(k) + Br(k) && \text{Closed – loop system} \end{aligned}$$

- Pole-placement problem Can we design a gain matrix K such that $A - BK$ has desired eigenvalues?
- Fact: Every uncontrollable open-loop eigenvalue is a closed-loop eigenvalue
- Assume
  - Complete controllability
  - Single input

Pole Placement using Companion Form

- Idea: Use state transformation $\hat{x} = Sx$ such that

$$\underbrace{\widehat{A} = SAS^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \widehat{B} = SB = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\text{controller canonical form}}$$

- Use feedback $u = -\widehat{K}\hat{x} = -[\hat{k}_n \;\; \hat{k}_{n-1} \;\; \cdots \;\; \hat{k}_1]\hat{x}$

$$\widehat{A} - \widehat{B}\widehat{K} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n - \hat{k}_n & -a_{n-1} - \hat{k}_{n-1} & -a_{n-2} - \hat{k}_{n-2} & \cdots & -a_1 - \hat{k}_1 \end{bmatrix}$$

Pole Placement using Companion Form (cont'd)

- Characteristic polynomial of A and $\widehat{A}$

$$z^n + a_1 z^{n-1} + \cdots + a_n$$

- Characteristic polynomial of $\widehat{A} - \widehat{B}\widehat{K}$

$$z^n + (a_1 + \hat{k}_1) z^{n-1} + \cdots + (a_n + \hat{k}_n)$$

- Desired characteristic polynomial

$$z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n$$

- Choose $\widehat{K} = [\alpha_n - a_n \;\; \alpha_{n-1} - a_{n-1} \;\; \cdots \;\; \alpha_1 - a_1]$

- Feedback in terms of original states

$$u = -\widehat{K}\hat{x} = -\underbrace{\widehat{K}S}_{K} x$$

- Ackermann's Formula:

$$K = e_n^{\mathrm{T}} \mathcal{C}^{-1} [A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I]$$

$$e_n^{\mathrm{T}} = [0 \; 0 \; \cdots \; 1]$$

Proof of Ackermann's Formula

- Define $s_1 = e_n^{\mathrm{T}} \mathcal{C}^{-1}$, and consider

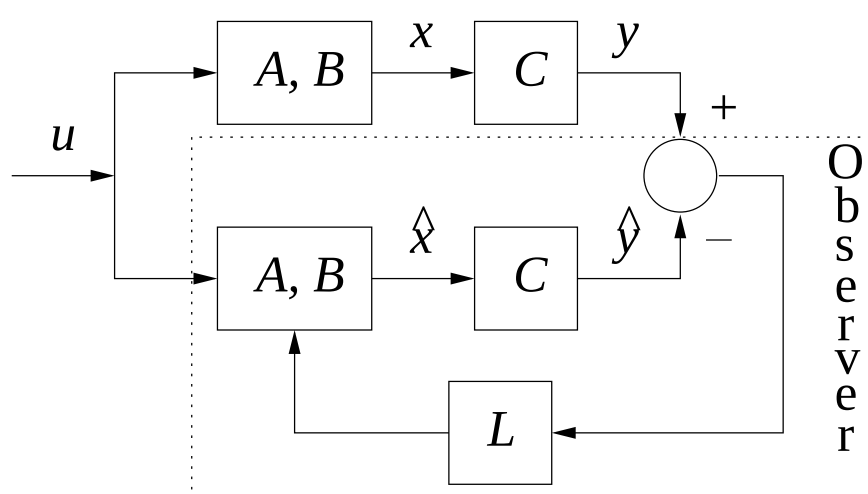
$$S = \begin{bmatrix} s_1 \\ s_1 A \\ \vdots \\ s_1 A^{n-1} \end{bmatrix}$$

- Claim: $SA = \widehat{A} S$
- Claim: S is invertible
- Claim: $SB = \widehat{B}$
  - S yields the transformation to the controller canonical form
- $K = \widehat{K} S$ yields Ackermann's formula
- Multi-input case: redundant degrees of freedom in choosing K
  - Can be used to assign eigenstructure

Estimation/Observation

- Online reconstruction of state vector from the output

$$\underbrace{\begin{array}{rcl} x(k+1) &=& Ax(k)+Bu(k) \\ y(k) &=& Cx(k) \end{array}}_{\text{Observable single-output system}}, \quad \underbrace{\begin{array}{rcl} \widehat{x}(k+1) &=& A\widehat{x}(k)+Bu(k)+\overbrace{L(y(k)-\widehat{y}(k))}^{\text{output injection}} \\ \widehat{y}(k) &=& C\hat{x}(k) \end{array}}_{\text{Luenberger observer}}$$



- Error dynamics

$$e(k+1) = (A-LC)e(k), \;\; e = x - \widehat{x}$$

Observer Design by Pole Placement

- Can we choose observer gain matrix L such that $A - LC$ is Schur?
- Yes, if the system is observable
  - (A, C) is observable iff $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is controllable
  - There exists a gain matrix K such that $A^{\mathrm{T}} - C^{\mathrm{T}}K$ has desired eigenvalues

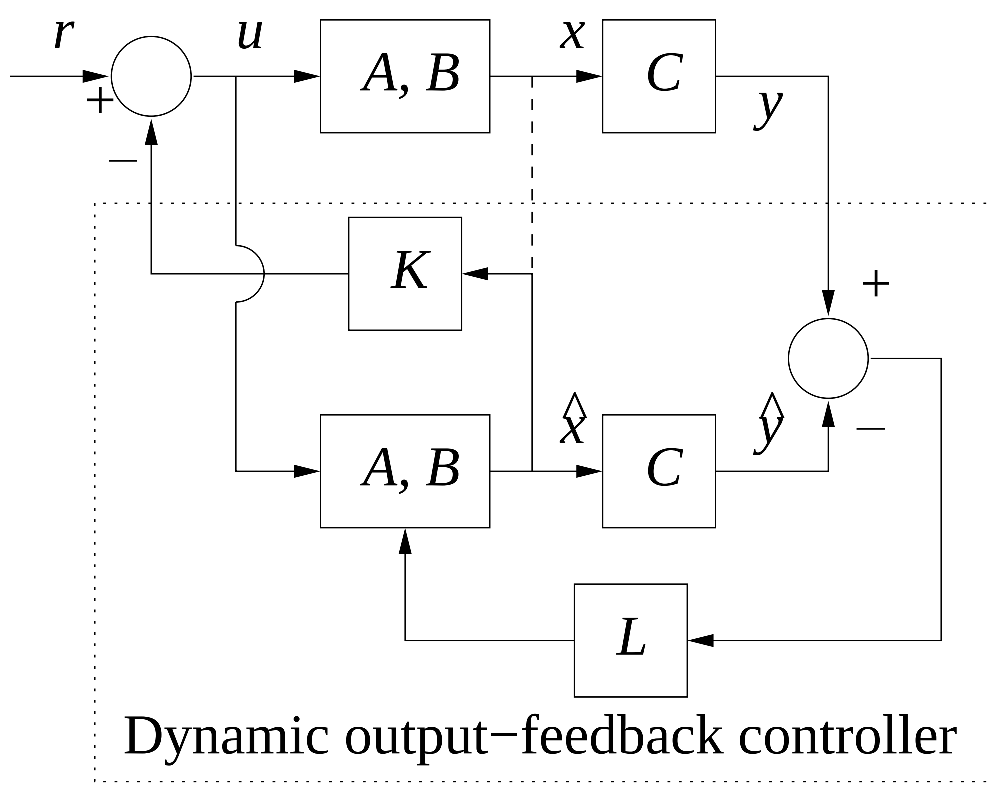
$$K = e_n^{\mathrm{T}}[C^{\mathrm{T}} \ C^{\mathrm{T}}A^{\mathrm{T}} \ \cdots \ C^{\mathrm{T}}A^{(n-1)\mathrm{T}}]^{-1}[A^{n\mathrm{T}} + \alpha_1 A^{(n-1)\mathrm{T}} + \cdots + \alpha_n I]$$

  - Letting $L = K^{\mathrm{T}}$, $A - LC$ has desired eigenvalues

$$L = [A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I]\mathcal{O}^{-1}e_n$$

Dynamic Output-Feedback Compensation

- Idea: In a full-state feedback controller, use state estimate generated by an observer in place of the actual state



Seperation Principle

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) & \widehat{x}(k+1) &= A\widehat{x}(k) + Bu(k) + L(y(k) - C\widehat{x}(k)) \\ y(k) &= Cx(k) & u(k) &= -K\hat{x}(k) + r(k) \end{aligned}$$

$$\underbrace{\begin{bmatrix} x(k+1) \\ \widehat{x}(k+1) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x(k) \\ \widehat{x}(k) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r(k)}_{\text{Closed}-\text{loop system}}$$

- Choose state vector as $[x^{\mathrm{T}}\ e^{\mathrm{T}}]^{\mathrm{T}}$, $e = x - \widehat{x}$

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r(k)$$

- Seperation Principle: Output-feedback controller can be obtained by combining an independently designed
  - Regulator that uses full state feedback with
  - An observer