Magnetic Precession and Product-of-Inertia Nutation Damping of Bias Momentum Satellites

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This paper is concerned with large-angle precession of Earth-pointing bias momentum satellites in circular orbits and small-angle precession in eccentric orbits. In the first part, ideal torque requirements are determined for precessing a momentum vector by a large yaw angle. It is then shown that a quasi-quadrant-orbit open-loop magnetic precession is feasible only when the spacecraft rotation for Earth pointing is halted. For this circumstance, general relationships are developed to size a pitch dipole using a tilted dipole model of the geomagnetic field and quasi-quadrant-orbit apart true anomalies whereas a pitch dipole reverses its polarity. The second part of the paper considers elliptic orbits. A pitch dipole sizing equation is developed for small-angle open-loop roll/yaw magnetic precession. This equation is then used for designing a closed-loop bang-bang precession scheme, working in concert with nutation damping via a roll/pitch product-of-inertia. A literal relationship is devised between the desired nutation damping coefficient (or time constant), roll/pitch product-of-inertia, and rate gain of a linear pitch controller.

I. Introduction

For efficient operation of Earth-pointing spacecraft, thermal radiators are placed so as to face the cold side of the orbit, and solar arrays are placed so as to face the warm side. Every few months, however, due to Earth's motion around the sun and due to Earth's oblateness, orbit planes of some spacecraft regress by nearly 5 months, however, due to Earth's motion around the sun and due to Earth's oblateness, orbit planes of some spacecraft regress by nearly 5 months. Hence the second objective of the paper is to size a pitch dipole for a small-angle closed-loop magnetic precession so that the spacecraft attains its nominal Earth-pointing attitude before diving again into the atmospheric pocket near perigee. The problem of small-angle precession arose, for instance, in the case of a multinational spacecraft AMPTE/IRM (active magnetospheric particle tracer explorers/ion release module) in a highly elliptic orbit, for which the electromagnet changed its polarity through time-tagged (that is, open-loop) commands. The closed-loop magnetic precession studied here is executed in conjunction with active nutation damping effected through a roll/pitch product-of-inertia and momentum wheel speed modulation. The development of a suitable relationship between various parameters involved in this process is the last objective of the paper.

Magnetic precession is a classical concept; to our knowledge, however, the available literature does not furnish a relationship to determine the pitch dipole strength for a general spacecraft orbit, general orientation of geomagnetic field, desired precession angle, and the number of orbits allocated for the precession; see the companion paper (Ref. 2) for a review of the past, significant contributions. Space limitations forbid including this review here, but certainly, formal techniques such as optimal control as well as others have been used successfully to design the magnetic controllers; however, because of their elaborate and rigorous mathematical framework, these techniques are unsuitable for preliminary conceptual verification and parametric studies. Simple sizing equations are therefore desired and hence developed in this paper for just this purpose. The ideal torque required for a 180-deg yaw precession of an Earth-pointing bias momentum satellite in a circular orbit is determined in Sec. II. This ideal torque profile about the roll and yaw axes is compared with the one producible by the electromagnets interacting with the geomagnetic field. An unexpected conclusion emerges that, due to the Earth-pointing requirement during precession, it appears infeasible to generate the ideal torque profile magnetically in an open-loop fashion. Section III then formulates a magnetic yaw precession after halting the once-per-orbit rotation rate w_o. This inertial yaw precession resembles the precession of spin-stabilized spacecraft, but the analysis presented here is general and fills a gap in the literature. Focusing on the elliptic orbits next, Refs. 1 and 3-5 do demonstrate precession control with a given pitch dipole, but none of these studies provides a way of sizing the dipole for precessing a momentum vector by a specified linear angle. A quasi-quarter-orbit, open-loop magnetic controller for eccentric orbits is therefore formulated in Sec. IV, and a literal relationship is developed to size the pitch dipole. For an autonomous implementation of this scheme, the onboard magnetic controller of Rajaram and Goel could be used. But inasmuch as they do not address damping of high-frequency
nutation excited inevitably, Sec. V presents a closed-loop, bang-bang, autonomous magnetic controller, working in conjunction with active nutation damping through a roll/pitch product-of-inertia and modulation of the momentum wheel speed with a proportional–integral–derivative (PID) pitch controller. The magnetic controller turns itself off judiciously when the spacecraft in the elliptic orbit reaches altitudes where the geomagnetic field is so feeble that it is strenuous to use it, provided the roll/yaw errors are also within deadband. The sizing equation developed in Sec. IV is employed to determine the required strength (ampere - meter$^2$) of the pitch dipole. A literal relationship is formulated between the specified nutation damping coefficient (or time constant), the roll/pitch product-of-inertia, and the rate gain of the pitch controller. The paper is finally concluded in Sec. VI.

II. Large-Angle Earth-Pointing Yaw Precession

Ideal Torque Requirement

By ideal, one means that the satellite’s own angular momentum is negligible compared to the bias momentum $h_b$ of the wheel residing in the satellite and spinning clockwise about the pitch or $y_b$ axis. Figure 1 illustrates the ideal yaw precession $\alpha_y(t)$ (i = time) of a bias momentum satellite about the nadir unit vector $e_3$. The unit vector triad $e_i, e_2, e_3$ is associated with the orbit frame $\mathbb{F}^c$: $x, y, z_c$ rotating clockwise at the rate $\omega_h(\omega_0 > 0)$ about the axis $y_c$ (Fig. 1); the inertial angular rate $\omega^c$ of the frame $\mathbb{F}^c$, expressed in $\mathbb{F}^i$, is therefore $\omega^c = -\omega_h e_2$. Furthermore, by definition, the ideal precession is accompanied with zero roll angle $\alpha_r = 0$ and zero pitch angle $\alpha_p = 0$ about the spacecraft-attached $x_b$ and $y_b$ axes, respectively. Consequently, the instantaneous angular momentum $h_i$, of the wheel, during precession, can be expressed in the frame $\mathbb{F}^c$ as

$$h_i(t) = h_i(-s\alpha_x e_1 + c\alpha_x e_2)$$

(1)

where $h_i < 0$ (clockwise about the unit vector $e_2$ at $t = 0$); $s = \sin(\cdot)$ and $c = \cos(\cdot)$. In view of the angular rate $\omega^c$, the inertial derivative of $h_i$ is

$$\dot{h}_i = -h_i\dot{\alpha}_x(-c\alpha_x e_1 + s\alpha_x e_2) - h_i\dot{\alpha}_x s\alpha_x e_3$$

(2)

where $\dot{\alpha}_x$ is the rate, not necessarily constant, at which the angular momentum is precessed. Here $h_i$ is the torque required for the specified yaw precession and, in view of Eq. (2), it has components not only in the roll pitch ($e_1, e_2$) plane but along the yaw axis $e_3$ as well owing to the orbit rate $\omega_0$ for Earth pointing. Additionally, Eq. (2) reveals that if, for example, a yaw precession of 7 radians is stipulated in three orbits (constrained by the battery storage capacity), then, on the average, $\omega_h/\omega_0 = 6$, and so the Earth-pointing requirement imposes a yaw control torque six times the average roll/pitch torque for the yaw precession. As we shall learn later, this renders an open-loop magnetic yaw precession infeasible.

In Fig. 1, the ideal (that is, $\alpha_r = 0 = \alpha_p$) spacecraft-attached frame $\mathbb{F}^c$: $x_b, y_b, z_b$ is such that when $\alpha_r = 0$, it coincides with the frame $\mathbb{F}^i$. In this ideal frame, the torque $h_i$ can be expressed as

$$h_i^T = -h_i[-\alpha_x 0 0]$$

(3)

where the superscript $T$ means transpose. Equation (3) states that the precession torque $-h_i\alpha_x$ acts about the $z_b$ axis, normal to both the instantaneous $h_i$ about the $y_b$ axis and the yaw precession axis $e_3$, as expected. In addition, the torque $-h_i\alpha_x\alpha_y$ is required about the $z_b$ axis for Earth-pointing of the spacecraft, as discussed above.

Yaw Precession Using Geomagnetic Field

Our next objective is to generate the torque vector (2) or (3) through the interaction of the electromagnets, residing in the spacecraft, with the geomagnetic field $\mathbf{B}$. Because the desired torque vector has roll ($\alpha_x$-) and yaw ($\alpha_y$-) components in the body-fixed frame, and because a pitch magnetic moment $M_{\alpha_p}$ along the $y_b$ axis generates a control torque $g$, having roll and yaw components

$$B_{c\alpha}^T = M_{\alpha_p}[B_{B0} - B_{B1}]^T$$

(4)

where $B_{c\alpha}^T = [B_{B0} B_{B0} B_{B3}]$ is the geomagnetic field in the body-fixed frame, it (the pitch dipole) appears to be an appropriate choice for producing the desired control torque.

To examine this possibility, we first note that, in the orbit frame $\mathbb{F}^c$, the field $\mathbf{B}$ is given by [cf. Eq. (A.12), McElvain$^1$]

$$B_{c\alpha}^T = \mu_m[s_{\alpha_n}c(\omega_0 t - \eta_m) - c\epsilon_m, 2s_{\epsilon_m}s(\omega_0 t - \eta_m)]$$

(5)

where $\mu_m = \mu_m/r$, $\eta_m = Earth's$ dipole strength (7.943 × 10$^{15}$ Wb m), $r = radius$ of the circular orbit; the angle $\epsilon_m$ is the instantaneous inclination of the orbit plane with the geomagnetic equator, and the angle $\eta_m$ is the angle in the orbit plane from the instantaneous ascending node of the orbit relative to Earth’s equator to the instantaneous ascending node of the orbit relative to the geomagnetic equator.$^7$ Because of rotation of the geomagnetic field with Earth and because of nodal regression, the angles $\epsilon_m$ and $\eta_m$ vary slowly with time and, in addition, they depend on the orbit inclination $i$ and the angle $y_m = (11.44 deg)$ between Earth’s magnetic dipole and the geographic north pole.$^{2,3}$

Knowing the field $\mathbf{B}$ in the orbit frame $\mathbb{F}^c$, it can now be written in the ideal, precessing body frame $\mathbb{F}^b$, wherefrom the components $B_{B0}$ and $B_{B3}$ are placed in Eq. (4), yielding

$$g_{c\alpha}^T = M_{\alpha_p}\mu_m$$

(6)

Pondering over the available control torque (6) and the desired precession torque (3), we arrive at the disappointing conclusion that, whereas the desired $b_1$ and $b_3$ torque components in Eq. (3) are independent, the two producible components in Eq. (6) are unfortunately not. Indeed, when we modulate the magnetic momentum $M_{\alpha_p}$ to generate the roll torque $-h_i\alpha_x$, (greater than zero, for $h_i < 0$ and $\alpha_x > 0$), the yaw component in Eq. (6) varies arbitrarily, irrespective to the desired profile $-h_i\alpha_x$. Worse yet, we fail to find any open-loop combination of modulation of the roll, pitch, and yaw dipoles, which would yield, even approximately, the three torque components in Eq. (3), including the zero pitch torque component. Consequently, it appears that the Earth-pointing requirement of the spacecraft during the yaw precession should be abandoned and an inertial precession should be considered instead. The Earth-pointing precession is possible by employing the reference maneuver scheme of Ref. 8, linear or bang-bang closed-loop orbit-averaged momentum removal schemes of Ref. 9, or the switching controller of Ref. 3, but such designs are beyond the intended scope of this paper and, besides, they will not yield a sizing equation for the pitch dipole; hence we proceed to formulate an inertial precession in the next section.

III. Inertial Yaw Precession

We now introduce a nonrotating orbit frame $\mathbb{F}^c$: $x', y', z'$, aligned with the frame $\mathbb{F}^c$ at $\omega_0 t = 0$—the ascending node of the orbit. (Its relationship with the standard quasi-inertial frame $\mathbf{I}$ at the ascending node is obvious: $I = -e_3, b = e_3', n = -e_3'$, where the unit vectors $e_3$, $e_3'$, and $e_3''$ are along the axes $x, y$, and $z$, respectively, and the unit vectors $I, b, n$ are defined in Sec. IV and Fig. 4.) We now redefine the ideal spacecraft-attached body frame $\mathbb{F}^b$ by aligning it initially with the frame $\mathbb{F}^c$ and processing it about the quasi-inertial axis $z'$ (opposite to the ascending node line $I$ and not rotating it about the axis $x', y'$, or $y_b$ for Earth pointing. This implies that the spacecraft’s Earth-pointing rotation $\omega_0$ is halted at the ascending node before commencing the precession about the yaw axis ($z_b$ or $z')$ at a certain switching true anomaly determined below.
Now, the torque required for the precession about the inertial axis $z'$ is obtained from Eq. (2) or Eq. (3) by substituting $\omega_0 = 0$ and replacing the frame $\mathcal{F}$ with $\mathcal{F'}$:
\[
\mathbf{h}_{\mathcal{F'}} = -h_2 \hat{\delta}_3 [\mathbf{c}] \mathbf{s} \alpha_{\mathcal{F}} [0 0]^T \tag{7}
\]
\[
\mathbf{h}_{\mathcal{F'}} = -h_2 \hat{\delta}_3 [1 0 0]^T \tag{8}
\]
On the other hand, the available, ideal ($\alpha_1 = 0 = \alpha_2$) control torque produced by a pitch dipole in the non-Earth-pointing frame $\mathcal{F}^p$ is
\[
\mathbf{h}_{\mathcal{F}^p} = M_{m2} \mathbf{\hat{h}}_m \left[ \frac{1}{2} s_\xi \epsilon_s (-s_\eta m + 3 \theta m) + s_\alpha_3 c_\xi \epsilon_m \right] \tag{9}
\]
where the angle $\theta m$ varies at twice the orbit frequency: $\theta m = 2 \omega_0 t - \eta m$. As a clarification, note that, since the frame $\mathcal{F}^p$, or now the body-fixed frame $\mathcal{F}^p$, does not rotate to maintain Earth pointing, it may be misleading to call the angles $\alpha_1, \alpha_2, \alpha_3$, respectively, roll, pitch, and yaw, but we continue to do so for simplicity. Moreover, the precession considered here is not general; it is restricted to that about the $z'$ axis only.

From the desired equality of the roll components in Eqs. (8) and (9), we infer that the sign of the dipole $M_{m2}$ must be reversed when the factor $[-\sin \eta m + 3 \sin(2 \omega_0 t - \eta m)]$ changes its sign [also see Eq. (8) and Fig. 3, Shigehara,5 and Eq. (5), Rajaram and Goel6]. From the multiple solutions of the equation $3 \sin(2 \omega_0 t - \eta m) = \sin \eta m$, the first two switching instants $t_1$ and $t_2$ are
\[
\omega_0 t_1 \triangleq \eta_1 = \frac{1}{2} (\eta m + \eta m^*) \tag{10}
\]
\[
\omega_0 t_2 \triangleq \eta_2 = \frac{1}{2} (\pi + \eta m - \eta m^*) \tag{11}
\]
where
\[
\eta m^* \triangleq \sin^{-1} \left( \frac{1}{3} \sin \eta m \right) \tag{12}
\]
Clearly, the gap ($\eta_2 - \eta_1$) is not a quarter orbit; rather,
\[
\eta_2 - \eta_1 = \frac{1}{2} \pi - \eta m^* \tag{13}
\]
$\eta m^*$ being the deviation from the quarter-orbit switching. From the periodicity of the function $\sin(2 \omega_0 t - \eta m)$, the following third and fourth switching angles are also arrived at:
\[
\omega_0 t_3 \triangleq \eta_3 = \pi + \eta_1 \tag{14}
\]
\[
\omega_0 t_4 \triangleq \eta_4 = \pi + \eta_2 \tag{15}
\]
where the gap ($\eta_3 - \eta_2$), in contrast with the gap ($\eta_2 - \eta_1$), is
\[
\eta_3 - \eta_2 = \frac{1}{2} \pi + \eta m^* \tag{16}
\]
Comparing Eq. (13) with Eq. (16), we conclude that the two consecutive polarities of the dipole $M_{m2}$ are of unequal duration, although $\eta_3 - \eta_1 = \pi = \eta_4 - \eta_2$. The fifth switching instant, wherefrom the modulation of $M_{m2}$ repeats itself, occurs at $\eta_5 = 2 \pi$.

We will now determine the strength of the dipole $M_{m2}$ required for a given yaw precession angle $\alpha_3$ in $N$ orbits. Since the roll torques from Eqs. (8) and (9) are equal, and since $-h_2 \hat{\delta}_3 > 0$, the instants of reversing the pitch dipole polarity during the yaw precession are obtained from

\[
M_{m2} = |M_{m2}| \frac{\sin(2 \omega_0 t)(-s_\eta m + 3 \theta m)}{(-2 \omega_0)} \tag{17}
\]
The precession rate $\dot{\alpha}_3$, and the precession angle $\Delta \alpha_3$ over one-half orbit are then found to be
\[
\dot{\alpha}_3 = \frac{\hat{\mu}_m}{|M_{m2}|} (-s_\xi, \epsilon_s (-s_\eta m + 3 \theta m)) \tag{18}
\]
\[
\Delta \alpha_3 = \hat{\mu}_m |M_{m2}| s_\xi s_\eta m s_\theta m + 3 c_\eta m \tag{19}
\]
Compared with Eq. (22) of Shigehara3 or Eq. (9) of Rajaram and Goel6 Eq. (19) is general. Since $N$ orbits have $2N$ halves, the total precession angle $\alpha_{3p}$ will be
\[
\alpha_{3p} = 2N \Delta \alpha_3 \tag{20}
\]
which yields the following magnetic dipole strength for the yaw precession angle $\alpha_{3p}$ in $N$ orbits:
\[
|M_{m2}| = -h_2 \omega_0 \dot{\alpha}_3 [2N \dot{\alpha}_3 s_\xi ((s_\eta m \sin \eta m + 3 \cos \eta m^*) - (-h_2 \omega_0)] \tag{21}
\]
For an untilted geomagnetic field, $\eta m = 0 = \eta m^*$ and $\epsilon_s = i_1$ and Eq. (21) then reduces to
\[
|M_{m2}| = -h_2 \omega_0 \dot{\alpha}_3 [6N \dot{\alpha}_3 s_\xi \omega_0 = -h_2 \omega_0 \Delta \alpha_3 (3 \dot{\alpha}_3 \sin i)] \tag{22}
\]
which is the same as Eq. (14) (Shigehara3) after resolving notational differences. For further details and illustration of Eq. (21), see Ref. 2.
The high-frequency, low-amplitude loops in the roll/yaw angles are nutations of the spacecraft's momentum axis. These do not grow to threaten the precession because the pitch dipole switches its polarity open loop and, insofar as nutation phase angle is concerned, randomly, without sensing the attitude and rates, every 24 or 26 min, much larger than the nutation period of 1.6 min. At the end of three orbits, although the pitch axis is precessed by 180 deg about the yaw axis, there are residual roll angle of nearly 5 deg and high-frequency nutations. They can both be eliminated by a linear magnetic controller designed in Ref. 10. As shown in Fig. 3, the roll torque $g_{33}$, bringing about the precession, is a rectified sinusoid with a magnetic controller designed in Ref. 10. As shown in Fig. 3, the roll torque $g_{33}$, bringing about the precession, is a rectified sinusoid with a magnetic controller designed in Ref. 10.

**Fig. 3** Roll torque $g_{33}$ for inertial yaw precession and accompanying yaw torque $g_{33}$.

In the standard with $I$ along the ascending node line, the unit vectors $I$, $b$, $n$ are defined such that when the true anomaly $\varpi$ measured from the ascending node line is zero, $\alpha_1 = 0$, $\alpha_2 = -n$, and $\alpha_3 = -l$. The frames $F^c$ and $F^v$ introduced earlier in Sec. III are not used here because they are defined for circular orbits, for which $F^c = F^v$. The small yaw angle $\alpha_2$ about the nadir vector $\alpha_3$ and the roll angle $\alpha_3$ about the unit vector $\alpha_1$ (when $\alpha_2 = 0$) lead to the spacecraft-fixed unit vector triad $b, b', b''$. The radial vector $l'$ of the frame $I' b' n$ (Fig. 4) subtends an angle $\eta$ with $I$ and its importance will be apparent shortly. The true anomaly measured from $l'$ is denoted $\nu; \nu = \eta - \eta_0$.

**IV. Small-Angle Open-Loop Precession in Elliptic Orbit**

Figure 4 depicts the coordinate frames required for the formulation of the problem at hand. The unit vectors $l, b, n$ constitute the standard $I b n$ frame, $F^{I b n}$, with $l$ along the ascending node line, $b$ normal to $I$ in the orbit plane, making an acute angle with the velocity vector, and the orbit normal $n = l \times b$. The rotating orbit frame $F^c$: $o_1 o_2 o_3$ is defined such that when the true anomaly $\eta$ measured from the ascending node line is zero, $o_1 = b$, $o_2 = -n$, and $o_3 = -l$. The frames $F^c$ and $F^v$ introduced earlier in Sec. III are not used here because they are defined for circular orbits, for which $F^c = F^v$. The small yaw angle $\alpha_2$ about the nadir vector $\alpha_3$ and the roll angle $\alpha_3$ about the unit vector $\alpha_1$ (when $\alpha_2 = 0$) lead to the spacecraft-fixed unit vector triad $b, b', b''$. The radial vector $l'$ of the frame $I' b' n$ (Fig. 4) subtends an angle $\eta$ with $I$ and its importance will be apparent shortly. The true anomaly measured from $l'$ is denoted $\nu; \nu = \eta - \eta_0$.

**Excess Angular Momentum**

Nominally, the angular momentum $h_i$ of the wheel equals $h_i \alpha_i (h_i < 0)$. In the presence of roll and yaw angles, however, $h_i$, is aligned with the instantaneous pitch axis $b_2$ (Fig. 4), and therefore, ignoring nutations, $h_i$ can be written in the orbit frame $F^v$ as

$$h_{i}^{F^v} = h_i [\alpha_3 \alpha_2 \alpha_1]^T$$

where $h_i = h_i b_i$ ($h_i < 0$). Because the orbit frame is rotating, the vector $h_i$ in Eq. (23) is not in a convenient form for developing a magnetic control policy. It is therefore expressed below (as is customary) in the quasi-inertial frame $F^{I b n}$,

$$h_{i}^{F^{I b n}} = -h_i [A_u c(\mu_u + \eta) + A_u s(\mu_u + \eta) I]$$

where $A_u$ is the amplitude and $\mu_u$ the phase of the deviation of the momentum vector from its intended position $-h_i n$ in the inertial frame:

$$A_u \triangleq (\alpha_2^2 + \alpha_3^2)^{1/2}, \quad \mu_u \triangleq \tan^{-1}(\alpha_3/\alpha_2)$$

Clearly, $A_u$ is the angle to reduce to zero using a pitch dipole. The control policy for doing so can be devised with ease if $h_i$ is now expressed in that quasi-inertial frame $F^{I b n}$, in which, at time $t = 0$, the resultant of the $l$ and $b$ components of $h_i$ is aligned with either the $l'$ or $b'$ unit vector in the orbit plane. It can be shown that, in the frame $F^{I b n}$, $h_i$ is

$$h_{i}^{F^{I b n}} = -h_i [A_u c(\mu_u + \eta - \eta_0) + A_u s(\mu_u + \eta - \eta_0)]$$

At time $t = 0$, the roll and yaw angles are denoted $\alpha_{0u}$ and $\alpha_{0v}$ and the true anomaly $\eta_0$. Therefore, to lay the initial, undesired component of the angular momentum entirely along the unit vector, say, $l'$, the angle $\eta_0$ must be

$$\eta_0 = \mu_{0u} + \eta_0$$

where $\mu_{0u} = \mu_u (t = 0)$. Then, the excess angular momentum at $t = 0$ is $-h_i A_u d(\dot{A}_{0u} = A_u$ at $t = 0$). Of course, instead of Eq. (27), the angle $\eta_0$ could be defined such that the undesired component of the initial angular momentum is along the unit vector $b'$, not $l'$.

**Sign Reversal Instants of Pitch Dipole**

Because the initial excess angular momentum is situated entirely along $l'$, it is helpful to express the magnetic control torque $g_2$ also in the frame $F^{I b n}$. Using the definition of $v = (\eta_0 - \eta_0)$ and the analogous definition $v_0 = \eta_0 - \eta$, the torque produced by a pitch electromagnet of strength $M_{2u}$ in the frame $F^{I b n}$ is found to be

$$g_{2}^{F^{I b n}} = (\mu_{0u}/2v^2)M_{2u}s_{u}[-c\nu_0 + 3s(2v - \nu_0)]$$

where $r$ is now the instantaneous radial distance of the spacecraft from Earth's mass center. To annihilate the excess momentum $-h_i A_u d l'$, the angular momentum desired from the magnetic torque is $h_i A_u d l'$. Therefore, the sign of $M_{2u}$ must be such that the $l'$ component of $g_2$, in Eq. (28), denoted $g_{2l}$, obeys the following condition at all times until the amplitude $A_u$ diminishes to zero:

$$\text{sgn}[M_{2u} s_{u}[-c\nu_0 + 3s(2v - \nu_0)] = \text{sgn}(h_i A_u)]$$

Clearly, the polarity of the pitch dipole $M_{2u}$ must be reversed when the sign of the factor [-] in Eq. (29) reverses; these switching instants are derived from

$$3s(2v - \nu_0) = \cos \nu_0$$

Also, one can just as easily arrive at the condition $3\sin(2v - \nu_0) = \sin \nu_0$, analogous to that in Sec. III instead of Eq. (30), by defining a unit vector $b'$, instead of $l'$, parallel to the given excess angular
momentum. In any event, for \( t > 0 \), the first switching point \( v_t \) obtained from Eq. (30) is
\[
v_t = \frac{1}{3} (v_m + \eta_m^4)
\]
(31)

where
\[
\eta_m^4 = \cos^{-1} \left( \frac{1}{3} v_m \right)
\]
(32)

The definitions of \( v_1 \) and \( \eta_m^4 \) are analogous to the earlier definitions (10) and (12) for large-angle yaw precession. The remaining three switching instants are determined from the multiple solutions of Eq. (30), arriving at
\[
v_2 = v_1 + (\pi - \eta_m^4) = \pi + \frac{1}{3} (v_m - \eta_m^4)
\]
(33)
\[
v_3 = v_1 + \pi
\]
(34)
\[
v_4 = v_1 + (\pi - \eta_m^4)
\]
(35)

Thus, as in the previous section, while the alternate switching points are one-half orbit apart \( (v_1 - v_3 = \pi = v_4 - v_2) \), the consecutive switching points are not necessarily one-quarter orbit apart unless \( v_m = \frac{2}{3} \pi \) and then \( \eta_m^4 = \frac{1}{2} \pi \).

Determination of Pitch Dipole Strength

Next we integrate the \( \Gamma \) component of the magnetic torque, \( g_{r'} \), from Eq. (28). Note that for a few orbit periods of magnetic control, \( J \Gamma' \dot{\gamma} \) can be regarded as inertial, and so the straightforward integration of \( \Gamma \) is legitimate. The radius \( r \) and the angle \( v \) in Eq. (28) are both time varying. To eliminate the time dependence of \( r \), let \( M_{m2} \) be the mean pitch dipole strength, yet to be determined, and let the instantaneous magnitude of \( M_{m2} \) be
\[
|M_{m2}| = r^3 \tilde{M}_{m2} / a^3
\]
(36)

where \( a \) is the semimajor axis of the elliptic orbit. This scheme of varying the strength of a pitch dipole in proportion to \( r^3 \) and using it over the entire orbit is not always acceptable, because for highly elliptic orbits it may impose an impractical pitch dipole strength near apogee. This limitation is removed in the next section, however. Proceeding with the modulation scheme (36) for \( \Gamma \), \( g_{r'} \) transforms to
\[
g_{r'} = \left( \mu_m / 2a^3 \right) \text{sgn}(M_{m2}) \tilde{M}_{m2} \sin \xi_m \left[ -\cos v_m + 3 \cos(2v - v_m) \right]
\]
(37)

where \( \text{sgn}(M_{m2}) \) is governed by the condition (29). Integrating the factor \([\cdot] \) in Eq. (37) over the entire orbit, accounting for the sign reversals of the dipole \( M_{m2} \) at the switching anomalies \( v_1, \ldots, v_4 \), and then equating the accrued momentum to the desired momentum \( h_{\text{ref}} = \mu_m a \tilde{M}_{m2} \), we obtain
\[
\pm 3 \mu_m \tilde{M}_{m2} \sin \xi_m \left( (2v_m - \pi) \cos \eta_m^4 - 2 \sin \eta_m^4 \right) + 3 e \cos \tilde{\eta} (\pi + \sin 2 \eta_m^4 - 2 \eta_m^4) / 4 = h_{\text{ref}} a \omega_{m0} \tilde{A}_0
\]
(38)

where \( \mu_m = \mu_m / a^3 \), replacing the earlier definition of \( \mu_m : \omega_{m0} = \) mean orbit rate, \( \tilde{\eta} = \eta_m + \eta - 2 \eta_m, \eta_{m0} = \) true anomaly of the perigee measured from the ascending node line, and \( e = \) orbit eccentricity. From the \pm in Eq. (38), that sign is retained, which renders the two sides compatible.

Table 2 Parameters for small-angle precession of bias momentum spacecraft in elliptic orbit

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altitude at perigee and apogee</td>
<td>120 and 6000 km, respectively</td>
</tr>
<tr>
<td>Orbit eccentricity</td>
<td>0.3115</td>
</tr>
<tr>
<td>Mean orbit rate ( \omega_{m0} ) and period ( \nu_0 )</td>
<td>0.0006885 rad/s and 152.086 min, respectively</td>
</tr>
<tr>
<td>True anomaly ( \eta_{m0} ) at perigee</td>
<td>95 deg</td>
</tr>
<tr>
<td>Geomagnetic field parameters ( t_{m0}, \xi_m )</td>
<td>0 deg</td>
</tr>
<tr>
<td>Initial conditions: roll ( \phi(0) ) and yaw ( \Theta(0) )</td>
<td>-11.44, 94.9 deg, respectively</td>
</tr>
<tr>
<td>True anomalies at sign reversals: ( \eta_1, \eta_2, \eta_3, \eta_4 )</td>
<td>-6.0, 0 deg, respectively</td>
</tr>
<tr>
<td></td>
<td>180, 0 deg, respectively</td>
</tr>
<tr>
<td>True anomalies at sign reversals: ( \eta_1, \eta_2, \eta_3, \eta_4 )</td>
<td>180, 168.56, 109.07 deg, respectively</td>
</tr>
<tr>
<td>True anomalies at sign reversals: ( \eta_1, \eta_2, \eta_3, \eta_4 )</td>
<td>138.8, 209.74, 318.81, 29.74 deg, respectively</td>
</tr>
</tbody>
</table>

Fig. 5 Magnetic precession of momentum vector: \( (\alpha_R^2 + \alpha_T^2)^{1/2} \) vs time.

Fig. 6 Quasi-quarter-orbit sign reversal and pitch dipole strength variation over one orbit.

Illustration

Consider a bias momentum spacecraft in an elliptic orbit now. Various parameters involved, including the switching true anomalies, are furnished in Table 2. The performance of this magnetic controller is illustrated in Figs. 5 and 6, obtained by integrating two first-order momentum equations about the inertially fixed axes \( \Gamma' \) and \( b' \) under only the magnetic torque \( g_{r'} \), Eq. (37), and no disturbance torque. The instantaneous true anomaly \( \eta(t) \) is obtained by integrating the rate \( \dot{\eta} \) for the elliptic orbit.

Figure 5 illustrates the precession of the momentum vector from an initial roll error of -6 deg to zero in one orbit period. The precession is accomplished by reversing the pitch dipole’s polarity according to the law (29) at the true anomalies \( \eta_i \) recorded in Table 2. Instead of one-quarter orbit, the reversal takes place after 70.94 and 109.06 deg, a deviation of 19.06 deg from 90 deg (cf. Ref. 6). The open-loop precession in Fig. 5 for an elliptic orbit resembles the closed-loop precession in Fig. 3 (Ref. 6) for spin-stabilized spacecraft in a circular orbit. The polarity reversal and the variation of the pitch dipole strength are shown in Fig. 6, where the maximum required strength is found to be nearly 200 A \( \cdot m^2 \), whereas the average strength \( M_{m2} = 85.85 A \cdot m^2 \).

The drawbacks of the above scheme are considerable, however. It is open-loop, so it may not be effective under disturbances; the dipole strength varies in proportion to \( r^3 \), so a pitch dipole of inordinate strength and hence that is costly and weighty may be required for highly elliptic orbits to utilize the weak magnetic field near apogee, and spacecraft nutations are ignored. Moreover, the scheme does not profit from the dipole’s full strength near perigee where a strong magnetic field is available anyway. Hence, a more practical and efficient control scheme without these shortcomings is desired. Since a linear magnetic controller with constant gains for a circular orbit
does not suit an elliptic orbit, a closed-loop bang-bang scheme along with an active nutation damper recommends itself for consideration.

V. Closed-Loop Bang-Bang Magnetic Precession and Active Nutation Damping

Although bang-bang magnetic momentum removal schemes have been used frequently for nominally zero-momentum spacecraft, they are not readily useful for magnetic precession of bias momentum spacecraft because the precession torque persistently excites high-frequency nutations that, if not damped, ultimately cause chattering of the pitch dipole. Therefore, a bang-bang magnetic precession controller must be accompanied with a nutation damper. With regard to nutation damping, the superiority of an active scheme over passive for dual-spin spacecraft, using nutation rate feedback and coupling of the pitch controller with the roll/yaw motion through the roll/pitch or yaw/pitch product-of-inertia, is well known. Consequently, a closed-loop bang-bang magnetic precession controller, working along with an active nutation controller, for bias momentum spacecraft in elliptic orbit is eminently suitable here and is shown in Fig. 7. The operation of this dual controller is essentially self-evident but its special features—the two deadbands and active nutation damping—are explained below. Also, *I* in Fig. 7 is the spacecraft’s central inertia dyadic.

Magnetic Precession Controller

The polarity of the pitch dipole is now determined according to the bang-bang control law

\[ M_{m2} = |M_{m2}| \text{sgn}(\Delta h \times B)_{y-component} \]  

(39)

where \( \Delta h \), the excess angular momentum obtained from Eq. (23), is

\[ \Delta h^{\text{ex}} = h_i[-\alpha_3 \ 0 \ \alpha_1]^{T} \]  

(40)

Substituting Eqs. (40) and (5) (wherein \( \mu_a \) and \( \text{off} \) for circular orbit are replaced with \( \mu_a/r_i^3 \) and true anomaly \( \eta \), respectively, for elliptic orbit) in the law (39), it can be shown that

\[ M_{m2} = |M_{m2}| \text{sgn} \left( \left( \mu_a/r_i^3 \right) h_i \sin \eta_i \times \left[ \alpha_1 \cos(\eta - \eta_m) + 2\alpha_3 \sin(\eta - \eta_m) \right] \right) \]  

(41)

The strength \( |M_{m2}| \) of the pitch dipole is set to be slightly larger than the mean strength \( M_{m2} \) [Eq. (38)]. The circular deadband in Fig. 7 precludes the use of the dipole when \( A_{\text{off}} = A_{\text{off}} = (\alpha_1, \alpha_3) \) if even if the magnetic field is strong; here, \( A_{\text{off}} \) is the maximum tolerable angle of deviation of the momentum vector \( h_i \) from the orbit normal, and \( \alpha_1, \alpha_3 \) (db = deadband) are the tolerable roll/yaw errors. If this inequality is not satisfied, the pitch dipole is turned on only if the surrounding magnetic field is strong enough to produce a substantial torque, that is, only if the spacecraft is at lower altitudes of the elliptic orbit. Quantitatively, if \( r_{\text{weak}} \) is the orbit radius beyond which the geomagnetic field is considered too feeble to be effective, Eq. (41) then suggests a deadband \( \Delta h_i \) (see Fig. 7) equal to \( \Delta h_i = (\mu_a/r_{\text{weak}}) h_i |\alpha_3| \sin i \), where the orbit inclination \( i \) is used because, for untitled geomagnetic dipole model, \( \xi = i \).

The pitch dipole is turned on with proper polarity if the argument of the sign function in Eq. (41) exceeds the deadband \( \Delta h_i \). The pitch dipole then generates roll and yaw control torques \( g_{31} \) and \( g_{33} \) according to Eq. (44).

Active Nutation Damping

The active nutation controller designed here does not use nutation rate feedback as in Ref. 11. Rather, it appears adequate to couple the roll/yaw motion with the pitch motion through a roll/pitch product of inertia and then damp the nutations through a standard PID pitch controller, as shown in Fig. 7. To accomplish this, Ref. 2 develops, using a root perturbation technique, the following parametric relationship between the desired nutation damping coefficient \( \zeta_n \) and the required roll/pitch product of inertia \( I_{12} \):

\[ \zeta_n \Omega_n^2 = r_nK_D/[2I_2(1 - r_{m})] \]  

(42)

where \( r_m \) is a nondimensional measure of the roll/pitch product-of-inertia \( I_{12} \) and \( \Omega_n^2 \) is the nutation frequency of the spacecraft with a nonzero \( r_{m} \), compared with the nutation frequency \( \Omega_m \) for zero product-of-inertia:

\[ r_m = I_{12}^2/(I_1I_2) < 1 \]  

(43)

\[ \Omega_n^2 = \Omega_m(1 - r_{m})^{-\frac{3}{2}} \]  

(44)

\[ \Omega_m = |h_i|/\sqrt{I_1I_3} \]  

(45)

In Eqs. (42–45) \( I_1, I_2, I_3 \) are the roll, pitch, and yaw moments of inertia, respectively, of the spacecraft. In Eq. (42), \( K_D \) is the derivative gain of the PID pitch controller:

\[ K_D = I_2(2\zeta \omega_{bw} + \lambda) \]  

(46)

\[ \lambda = 5\zeta \omega_{bw} \]  

(47)

\( \omega_{bw} \) being the bandwidth, \( \zeta \) the damping coefficient (0.707, typically), and \(-\lambda \) the real root of the pitch controller. For the specified parameters \( \zeta, \Omega_n, K_D \) and the pitch inertia \( I_2 \), Eq. (42) specifies the required product-of-inertia parameter \( r_{m} \). It is reassuring to note that the nutation damping time constant [Eq. (14)] is for dual-spin spacecraft reduces to \( \zeta_n \Omega_n^2 \) [Eq. (42)] when, in the former equation, the gain \( K_D \) and the phase (−90 deg) of the pitch controller at the frequency \( \Omega_n \) are substituted.

Fig. 7 Bang-bang magnetic precession controller and active nutation damping via roll/pitch product-of-inertia.
Consider a small spacecraft designed for aeronomy traversing an elliptic orbit. Its relevant parameters are recorded in Table 2 except $h_s$, which is now $-81.35 \text{ N} \cdot \text{m} \cdot \text{s}$; the following additional parameters also apply:

$$I_1, I_2, I_3 = 81.78, 76.09, 60.25 \text{ kg} \cdot \text{m}^2$$
$$\alpha_m = 0.3469$$
$$\omega_m/\omega_0 = 16, \zeta = 0.707$$
$$I_{12} = 46.46 \text{ kg} \cdot \text{m}^2$$
$$\omega_{n}/\omega_0 = 1682.68$$
$$\Omega_{n}/\omega_0 = 2082.66$$

$$\Omega_n/\omega_m = 105.16, 2\pi/\Omega_n = 4.38 \text{ s (nutation period)}$$

where $\alpha_m$ and $I_{12}$ correspond to the nutation damping coefficient $\zeta = 0.01$. Reflecting on the values of $\Omega_n/\omega_0, \Omega_{n}/\omega_m$, and $\omega_m/\omega_0$, the assumptions in the analysis that the wheel angular momentum $\alpha_m$ is far greater than the satellite's own momentum and the nutation frequency is much greater than the pitch controller frequency are both justified. The sample period for the pitch controller is selected to be 0.25 s and that for the precession controller is set at 1 s. Because of low perigee altitude (120 km), the spacecraft experiences a significant atmospheric torque near perigee, shown in Fig. 8, where $t = 0$ corresponds to the true anomaly $\eta = -45$ deg. Since the orbital period is 152 min, this repetitive torque is quasi-impulsive, and the performance of the above controller over 480 min (more than three orbits) is shown in Figs. 9–12.

The amplitude $A_{a,off}$ is set at 0.2 deg, and the orbit radius $r_{orbit}$ corresponds to the true anomaly $\eta = \pm 135$ deg; the deadband $A_{d,off}$ is calculated accordingly. The roll and yaw magnetic control torque and the corresponding bang-bang profile of the pitch dipole with some high-frequency switching are shown in Fig. 9. The positive yaw atmospheric torque (Fig. 8) produces a significant negative roll precession in the beginning (Fig. 10), which in turn activates both the bang-bang magnetic controller (Fig. 9) and the PID pitch controller (Fig. 11). Subsequent motion is very coupled and it is clear from Fig. 9 that the magnets are on for long periods (60 min approximately) and turned off after nearly 65 min when the deadband conditions are satisfied. After approximately $t = 100$ min, the magnets are turned on and off several times relatively rapidly near the $A_{a,off}$ circle, and the nutations that are caused by the magnets are well damped by the speed modulation of the wheel, governed by the PID controller (Fig. 11). Figure 12 exhibits well-damped rate profiles, with a time constant of $1/\zeta \Omega_n = 1.16$ min. The attitude and rate determination aspect of the above scheme is sketched in Ref. 2.
VI. Concluding Remarks

The general sizing equation for a pitch dipole developed in the preceding for $\pi$ radian yaw precession of bias momentum satellites in circular orbits is useful for quick studies. Such yaw precessions, however, are inevitably accompanied with high-frequency nutations and substantial roll motion, both within linear range. After completing the yaw precession, these undesired motions can be eliminated using a linear magnetic controller. Moreover, the Earth-pointing rate about the pitch axis, halted before commencing the precession, must be reestablished after precession, and any large deviation from the Earth-pointing attitude must be corrected for by modulating the pitch momentum wheel speed. For elliptic orbits, the pitch dipole sizing equation for small angle roll/yaw precession is useful for determining the strength (in ampere · meter$^2$) required for a closed-loop, bang-bang magnetic precession. One such closed-loop controller, working together with nutation damping via a roll/pitch product-of-inertia and a pitch controller, and its performance under periodic, nearly impulsive, atmospheric torque around the perigee is presented. In a continuation of this study but not reported here due to space limitations, the performance of this controller is found to be very satisfactory even under realistic noisy measurements from horizon sensors and sun sensors.

References