Roll/Pitch Determination with Scanning Horizon Sensors: Oblateness and Altitude Corrections

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Analytical models are developed to relate horizon sensor locations, measurements, and their mounting geometry with roll/pitch attitude of Earth-pointing spacecraft, negotiating circular orbits around an oblate Earth. Two arrangements of a pair of horizon sensors are considered: left and right and aft and forward of the velocity vector, tilted in the pitch–yaw or roll–yaw plane, respectively. The corresponding roll/pitch oblateness corrections are formulated that involve 1) noncircularity of the Earth disk seen from space, 2) slight changes in azimuth angles of the horizon crossing points, and 3) scan angles of the sensors to travel the Earth disk. These corrections are compared with the closed-form roll/pitch components of the angle between geodetic and geocentric normals at the instantaneous location of the spacecraft—an alternate approach, also developed in the paper, to determine the corrections. The two approaches yield nearly the same result if a spacecraft is equipped with a pair of sensors (or four diagonal sensors). In that instance, the use onboard of the closed-form expressions of oblateness corrections, independent of the sensor geometry and location, suggests itself. But if due to cost, weight, or configuration constraints, only one Earth sensor is used, the corrections will then depend on the sensor geometry and location and may be determined for a given orbit before flight and stored numerically to minimize the processing load on the flight computer. The attitude corrections in the roll/pitch measurements, corresponding to the use of one or two sensors and arising from the spacecraft’s altitude variations, are also formulated and illustrated in the paper.

I. Introduction

HORIZON sensors have been used for nearly three decades to measure roll and pitch attitude of Earth-pointing and spin-stabilized satellites; see, for example, Ref. 1 for TIROS satellites and Ref. 2 for more recent satellites wherein a horizon sensor is paired with a fixed head star tracker, fine sun sensor, digital sun sensor, or a three-axis magnetometer, depending on the attitude determination accuracy desired. In this long span, great strides have been made in horizon sensing techniques, resulting in significantly improved accuracy in the attitude determination.1–7 The accuracy of the measurements is known to depend on random instrumental errors and on the correction of such deterministic errors as 1) seasonal variation in Earth’s radiance (not including mesoscale weather patterns or sudden polar stratospheric warmings)8, 9) Earth’s oblateness and spacecraft altitude variations, 3) ambient temperature, 4) spin period change, and 5) misalignment and biases; see, for instance, Alex and Shrivastava6 and Space Sciences2 for the contribution of each source to the attitude error budget (0.454 deg rms, total) and how the sum total is reduced to nearly one-eighth amplitude (0.06 deg rms) by onboard corrections. Of the five deterministic error sources just mentioned, this paper focuses on the attitude errors caused by oblateness and spacecraft altitude variations and onboard corrections to eliminate these errors. The paper is limited to the scanning sensors on Earth-pointing satellites; static Earth sensors and spin-stabilized satellites are not considered.

In the literature, Earth’s oblateness has been accounted for in two ways: 1) by determining the crossing points of the sensor scanpath on the oblate Earth disk (Liu,9 Ohtakay and Havens,9 and Collini10) and 2) by determining the deviation of the geodetic normal from the geocentric normal (Lebsock and Eterno11). The first approach is indispensable when only one scanning sensor is used for the attitude determination. The second approach, on the other hand, is an outgrowth of the classical Earth’s oblateness modeling technique used for inertial navigation of aircraft (Britting,12 Secs. 3.4, 4.1, and 4.2). It is likely that both approaches have been modeled, analyzed, and compared in company reports to determine accurately the roll and pitch attitudes, but the treatment in the open literature is fragmentary and scattered. Dissatisfied with thready details available, we present in this paper the models of roll/pitch attitude determination using scanning horizon sensors—solo or paired in the pitch–yaw or the roll–yaw plane. The contents of the paper are as follows. Section II deals with roll and pitch attitude determination with a pair of scanning horizon sensors on an Earth-pointing satellite, orbiting a spherical Earth. Basic relationships are established between the measurements from the horizon sensors left and right of the forward velocity, their tilt and cone angle, and the roll/pitch attitude of the spacecraft. Expressions of the azimuth angles of the four crossing points on a circular Earth disk are also formulated. The oblate Earth is considered in Sec. III. A simplified expression of noncircularity of the Earth disk, valid up to the first order of the flatness factor, is developed to arrive at the roll/pitch oblateness corrections. Deviation of the geodetic normal from the geocentric normal along the spacecraft orbit and its roll/pitch components are formulated in Sec. IV and compared numerically with the roll/pitch corrections developed in Sec. III. Section V presents the roll/pitch attitude determination and oblateness corrections for the sensors aft and forward of the velocity vector. The roll/pitch corrections arising from the deviation of the spacecraft from its nominal constant altitude for both pairs of sensors are formulated in Sec. VI. Finally, the paper is summed up in Sec. VII. To bring more cohesion in the technical literature, the literal relationships and the numerical results in this paper are compared with those openly available.

II. Sensors in Pitch–Yaw Plane: Spherical Earth

Throughout the paper, we are concerned with small roll and pitch angles of Earth-pointing spacecraft in a circular orbit, and therefore we shall treat their determination separately. Figure 1 depicts the geometry of a scanning horizon sensor on the right of the spacecraft’s forward motion. For a circular orbit, a local vertical local horizontal frame $F'$ is customarily defined to consist of a unit vector $e_1$ along the velocity vector, $e_2$ along the nadir from the spacecraft mass center to Earth’s mass center, and a unit vector $e_3$ opposite to the orbit normal and equal to $e_1 \times e_1$. The spacecraft rotates clockwise (cw) once per orbit about the axis $e_2$ with an angular rate of $\omega_0$ ($\omega_0 > 0$). When the roll, pitch, and yaw attitude angles are all zero, the body-fixed unit vector triad $b_1, b_2, b_3$ is aligned with the triad $e_1, e_2, e_3$. The spin axis $s_2$ of the scanning Earth sensor is tilted by an angle $\xi$ (10–25 deg, generally) from the pitch axis $e_2$ in the
Fig. 1 Geometry and scan path: right tilted sensor.

pitch–yaw plane, as shown. When there is a nonzero roll or yaw angle, the angle \( \xi \) will be measured from the body-fixed axis \( b_2 \), not \( c_2 \). The optic axis \( O \) of the sensor makes a semi-cone angle \( \delta_3 \) (30–60 deg) with the spin axis and forms a cone in space, intersecting Earth. Usually, the sensor scans counterclockwise (ccw) about the axis \( s_2 \) and the associated angular momentum is negligible (say, \( 10^{-4} \) N·m·s) to preclude its influence on the spacecraft dynamics. However, for bias momentum satellites, the sensor could be mounted on a momentum wheel, forming a Scanwheel\textsuperscript{®} (manufactured by Ithaco Space Systems). The associated angular momentum about the axis \( s_2 \) is then dynamically significant, and it is usually clockwise so as to add to the satellite’s angular momentum arising from the once-per-orbit clockwise rotation about the axis \( c_2 \) and render a more stable satellite. (In that circumstance, there must be a left sensor, also tilted by an angle \( \xi \), so that the net bias momentum is along the pitch axis.) The roll and pitch measurements and the oblateness corrections do not depend on the sense of rotation of the sensor; however, the space-to-Earth crossing point \( O_{QR} \) and the Earth-to-space crossing point \( O_{IR} \), shown in Fig. 1 for counterclockwise scan motion, will interchange for the clockwise scan motion, and the form of the roll and pitch equations and oblate corrections will change correspondingly. These changes will be illustrated subsequently. Continuing with Fig. 1 for now, the Earth disk visible from the satellite at an altitude \( h \) is of angular radius \( \rho_0 \) defined by

\[
\sin \rho_0 = R_\oplus/(R_\oplus + h) \tag{1}
\]

where the subscript \( \oplus \) denotes a quantity pertaining to a spherical Earth of radius \( R_\oplus \). The scan angle width (\( \theta_{w0} \) to \( \theta_{wR} \)) of the spherical Earth about the axis \( s_2 \), for zero roll and pitch angles, is denoted \( 2\theta_{w0} \). The tangency of the optic axis \( O \) at the transit points \( O_{QR} \) and \( O_{IR} \) yields

\[
\cos \rho_0 = \hat{O} \cdot c_3 \tag{2}
\]

where \( \hat{O} \) is the unit vector along the optic axis \( O \). Expressing \( \hat{O} \) at the transit points in the frame \( \mathcal{F}_c \), the semiscan angle \( \theta_{w0} \) for the spherical Earth can be shown to be governed by

\[
c \rho_{0} = c \delta_{3} s \xi + s \delta_{3} c \xi c \theta_{w0} \tag{3}
\]

where the common practice of writing \( s \cdot \) for \( \sin(\cdot) \) and \( c \cdot \) for \( \cos(\cdot) \) is adopted.

Roll/Pitch Determination: Right Sensor

For a positive roll angle \( \alpha_1 \), the scanwidth \( 2\theta_w \) (from \( O_{QR} \) to \( O_{IR} \)) of the Earth disk, for the right sensor, is larger than \( 2\theta_{w0} \), corresponding to the zero roll and pitch attitude of the spacecraft. For small roll and pitch angles, the roll angle \( \alpha_1 \) is in the plane \( c_2 c_3 \), and therefore, for nonzero \( \alpha_1 \), Eq. (3) modifies to

\[
c \rho_{0} = c \delta_{3} s (\xi + \alpha_1) + s \delta_{3} c (\xi + \alpha_1) c \theta_{w} \tag{4}
\]

valid for both crossing points \( O_{QR} \) and \( O_{IR} \). For \( \alpha_1 \ll 1 \), the semiscan angle \( \theta_{w0} \) will be only slightly different from \( \theta_{w0} \), and therefore we can write

\[
\theta_{w} = \theta_{w0} + \Delta_{w} \tag{5}
\]

where \( | \Delta_{w} | \ll \theta_{w0} \). For small \( \alpha_1 \) and \( \Delta_{w} \), Eq. (4) then leads to the linear relationship

\[
\alpha_1 = K \Delta_{w} = K (\theta_{w} - \theta_{w0}) \tag{6}
\]

where the nominal slope \( K \) is governed by the nominal parameters \( \xi, \delta_3, \) and \( \theta_{w0} \) (and therefore the altitude \( h \)) as follows:

\[
K = \sin \theta_{w0} / (c \delta_3 - \tan \xi \cos \theta_{w0}) \tag{7}
\]

For zero pitch angle, when the reference mark on the scanning sensor aligns itself momentarily with the axis \( s_3 \) in the pitch–yaw plane
(Fig. 1), the scanpath \( O_{0R} O_{1R} \) (equal to \( 2\phi_w \)) is divided equally. However, for nonzero pitch angle the scanpath from \( O_{0R} \) to the reference mark and from the reference mark to \( O_{1R} \) will be, respectively, \( \theta_{w0R} \) and \( \theta_{w1R} \) (\( \theta_{w0R} > \theta_{w1R} \) \foreignlanguage{en}{for a positive pitch angle}). For a nonzero pitch angle, then, Eq. (6) changes to

\[
\alpha_1 = \frac{1}{2} K (\theta_{w0R} + \theta_{w1R} - 2\theta_{w0B})
\]  

(8)

where the scan angles \( \theta_{w0R} \) and \( \theta_{w1R} \) are furnished by the sensor and \( \theta_{w0B} \) is known. The scanwidth \( \theta_{w0R} + \theta_{w1R} \) is the so-called chord.

For determining the pitch angle \( \alpha_2 \) about the unit vector \( e_2 \), the unit vector \( O \) at the transit point \( O_{0R} \) or \( O_{1R} \) is first expressed in the frame \( S_1 S_2 S_3 \) (Fig. 1). It is then transformed to the spacecraft-attached frame \( b_1 b_2 b_3 \) through the angle \( \xi \) and then to the local vertical local horizontal frame \( F \) through the pitch angle \( \alpha_2 \). Applying Eq. (2) then yields, at \( O_{1R} \),

\[
c_{\alpha_2} = c_{\delta_1} s_2 + s_{\delta_1} c_2 c_{\theta_{w1R}} - \alpha_2 c_{\delta_1} s_2 \theta_{w1R}
\]  

(9)

The equation for the in-crossing point \( O_{1R} \) is obtained from Eq. (9) by replacing \( \theta_{w1R} \) with \(-\theta_{w1R}\). Invoking the small-angle approximation similar to Eq. (5) and ignoring the second-order products, for the ccw scan, Eq. (9) and its companion for \( \alpha_1 \) lead to

\[
\alpha_2 = \frac{1}{2} (\theta_{w0R} - \theta_{w1R}) \cos \xi
\]  

(10)

where, as stated earlier, \( \theta_{w0R} > \theta_{w1R} \) for \( \alpha_2 > 0 \). If the sensor were to scan Earth clockwise (as in the case of a Scanwheel), the transit points \( O_{0R} \) and \( O_{1R} \) in Fig. 1 will interchange and Eq. (10) would then transform, for the cw scan, to

\[
\alpha_2 = \frac{1}{2} (\theta_{w1R} - \theta_{w0R}) \cos \xi
\]  

(11)

For \( \alpha_2 > 0 \), we would now have \( \theta_{w1R} > \theta_{w0R} \). The roll and pitch angles thus remain the same regardless of the scan direction of the sensor.

In the literature, the pitch angle \( \alpha_2 \) is often expressed in terms of a phase angle. Since the chord \( 2\phi_w \equiv \theta_{w0R} + \theta_{w1R} \), Eqs. (10) and (11) can be rewritten in the form

\[
\alpha_2 = (\theta_{w0R} - \theta_{w1R}) \cos \xi = (\theta_w - \theta_{w1R}) \cos \xi
\]  

(12a)

\[
\alpha_2 = (\theta_{w0R} - \theta_{w1R}) \cos \xi = (\theta_{w0R} - \theta_{w1R}) \cos \xi
\]  

(12b)

for ccw and cw scanning, respectively, where \( \theta_{w0R} - \theta_w, \theta_{w0R} - \theta_{w1R} \), etc., are the phase angles.

The pitch angle \( \alpha_2 \) is not sensitive to the altitude variations of the spacecraft because it (the pitch) depends on the measured angles \( \theta_{w0R} \) and \( \theta_{w1R} \) and the tilt angle \( \xi \). The accuracy of the roll angle (Eq. (8)), on the other hand, is influenced by the altitude variations, for both the nominal angle \( \theta_w \) and the slope \( K \) are preselected for a nominal altitude. If the true altitude is known, the parameters \( K \) and \( \theta_{w0R} \) can be updated in flight as the altitude varies; otherwise, a well-known alternative is to use another sensor on the left of the velocity vector (the \(-c_2 \) side). This configuration is considered next.

**Roll/Pitch Determination: Left Sensor**

Referring to Fig. 2, the scan cone axis \(-s_2 \) of the left sensor is tilted by the angle \( \xi \) from the axis \(-c_2 \) (the negative pitch axis, or the orbit normal). The sensor scans with the same counterclockwise angular velocity \( \omega_0 \) about the axis \( s_2 \) as before, except that the \( s_2 \) axes of the two sensors are not parallel. Instead, they are arranged symmetrically about the pitch axis \( b_2 \) (or \( c_2 \) if the roll/pitch/yaw angles are all zero) in the pitch-yaw plane such that the net angular momentum is about the pitch axis \( b_1 \). In the case of a Scanwheel, which has a significant angular momentum by design, the angular velocity \( \omega_0 \) about the axis \( s_2 \) is usually clockwise; but Fig. 2 pertains to a scenario in which the angular momentum of the sensor is negligible and it spins counterclockwise about the axis \( s_2 \). The corresponding transit points space-to-Earth, \( O_{0L} \), and Earth-to-space, \( O_{1L} \), and the scanpath on the Earth disk are also shown in Fig. 2. The scan angles from \( O_{0L} \) to the reference mark aligned with \( s_2 \) and from the reference mark to \( O_{1L} \) are \( \theta_{w0L} \) and \( \theta_{w1L} \), respectively.

![Fig. 2 Geometry and scan path: left tilted sensor.](image-url)
As a side comment, note that, by a still different arrangement, the sensors could scan Earth in opposite directions; see Sec. V.

Equation (3), derived for the right sensor for zero roll/pitch/yaw angles, is valid even for the left sensor. But as the roll angle $\alpha_1$ is now opposite to the tilt angle $\xi$ in Fig. 2, the angle $\alpha_1$ in Eq. (4) is replaced by $-\alpha_1$. Therefore,

$$c_{\phi_{\theta}} = c_\xi s_{\xi} - \alpha_1 + s_\xi c_\xi (\xi - \alpha_1) c_{\theta w_{1L}} = 0, 1 \quad (13a)$$

valid for both in-crossing and out-crossing points. On the other hand, the tangency equation (2) applied to the left out-crossing point, with the spacecraft at a pitch angle $\alpha_2$, is, at $O_{1L}$,

$$c_{\phi_{\theta}} = c_\xi s_{\xi} + s_\xi c_\xi c_{\theta w_{1L}} - \alpha_2 s_\xi s_{\theta w_{1L}} \quad (13b)$$

formally the same as Eq. (9) for the right out-crossing point, because if the right sensor’s optic axis were located at $-\xi$ and $(-180 - \delta, \delta)$ deg, it would occupy the left sensor’s optic axis position. Substituting these angles in place of $\xi$ and $\delta$, and $\theta w_{1L}$ in place of $\theta w_{1R}$ in Eq. (9), Eq. (13b) follows. For positive roll angle, the scanwidth in Eq. (9), Eq. (13b) diminishes (see Fig. 2, Ref. 6). Following the previous linear analysis for the right sensor and using Eqs. (13a), (13b), and its counterpart for the in-crossing point, we arrive at these roll and pitch measurement equations:

$$\alpha_1 = \frac{-1}{2} K (\theta_{o_{1L}} + \theta_{w_{1L}} - 2\theta_{\omega_{1L}}) \quad (14)$$

and

$$\alpha_2 = \frac{1}{2} \cos \xi (\theta_{o_{1L}} - \theta_{w_{1L}}) \quad (15a)$$

for counterclockwise scanning and

$$\alpha_2 = \frac{1}{2} \cos \xi (\theta_{o_{1L}} - \theta_{w_{1L}}) \quad (15b)$$

for clockwise scanning. The comments made earlier for the right sensor regarding the corrections for altitude variations are valid for the left sensor as well.

Roll/Pitch Determination: Both Sensors

To eliminate the dependence of the roll angle on the altitude knowledge in Eqs. (8) and (14), these two measurements from the two sensors are combined and averaged. Likewise, averaging the pitch angles from the previous equations, the following roll and pitch equations are found:

$$\alpha_1 = \frac{1}{2} K (\theta_{o_{1R}} + \theta_{w_{1R}} - \theta_{o_{1L}} - \theta_{w_{1L}}) \quad (16)$$

and

$$\alpha_2 = \frac{1}{2} \cos \xi (\theta_{o_{1R}} - \theta_{w_{1R}} + \theta_{o_{1L}} - \theta_{w_{1L}}) \quad (17a)$$

for counterclockwise scanning and

$$\alpha_2 = \frac{1}{2} \cos \xi (\theta_{o_{1R}} - \theta_{w_{1R}} + \theta_{o_{1L}} - \theta_{w_{1L}}) \quad (17b)$$

for clockwise scanning. The dependence of the roll angle on the altitude is not completely eliminated, though, since the slope $K$ in Eq. (16) is a function of the semiscan angle $\theta_{w_{1L}}$, which in turn depends on the altitude. But as we shall see later, that is a comparatively weak dependence. Furthermore, not surprisingly, Eqs. (16) and (17b) match with the roll and pitch measurement equation (3) by Ohtakay and Havens’ for the SEASAT-A bias momentum satellite, establishing thereby the literal values of their constants $k_r$, and $k_p$: $k_r = 4/K$ and $k_p = 4/\cos \xi$. And finally, Eqs. (16) and (17b) can be interpreted in terms of phase and chord associated with each scanpath.

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**Fig. 3 Four horizon crossing points for spherical Earth and positive roll angle: sensors scanning counterclockwise about pitch axis.**

Points of Horizon Crossings

Anticipating our later needs while calculating the oblateness corrections, we now determine the azimuths of the four crossing points on the circular Earth disk (see Fig. 3). The azimuth angle, denoted by $\psi$, is measured from the local east $E$ about the local vertical $u$, at the instantaneous location of the spacecraft in the orbit or, equivalently (for a spherical Earth), at the subsatellite point. For this purpose, let $i$ be the orbit inclination angle and $a_{1R}$ the spacecraft’s true anomaly measured from the ascending node line, which is at an angle $\Omega_{1R}$ from the vernal equinox. Then, the longitude $\phi$ measured from the vernal equinox and latitude $\lambda$ of the subsatellite point, both angles in the geocentric inertial frame, and the heading angle $\gamma$ of the spacecraft’s velocity (the unit vector $c_1$), measured ccw from the local east (see Fig. 3), are all given by the following:

Longitude

$$\phi = \Omega_{1R} + \tan^{-1}(\cos i \tan a_{1R}) \quad (18a)$$

Latitude

$$\lambda = \sin^{-1}(\sin i \sin a_{1R}) \quad (18b)$$

Heading angle

$$\gamma = \tan^{-1}(\tan i \cos a_{1R}) \quad (18c)$$

For zero roll and zero pitch angle and counterclockwise scan, expressing the unit vector $\hat{O}$ at the out-crossing point of the right sensor in the frame $F_1$: $c_1, c_2, c_3$, and, using Eq. (19), we arrive at

$$\tan \psi'_{1R} = (-c_\xi c_\xi + s_\xi s_\xi c_{\theta w_{1R}})/(s_\xi s_\xi s_{\theta w_{1R}}) \quad (20a)$$

where, as before, the subscript $1R$ denotes the out-crossing point. The expression for the in-crossing azimuth $\psi'_{1L}$ for the right sensor is obtained from Eq. (20a) by replacing the semiscan angle $\theta_{w_{1R}}$ with $-\theta_{w_{1L}}$, yielding

$$\tan \psi'_{1L} = (-c_\xi c_\xi + s_\xi s_\xi s_{\theta w_{1R}})/(s_\xi s_\xi s_{\theta w_{1L}}) \quad (20b)$$

The expressions for the in-crossing and out-crossing azimuth angles for the left sensor, $\psi'_{1L}$ and $\psi'_{1R}$, derived likewise, are as follows:

$$\tan \psi'_{1L} = (c_\xi c_\xi - s_\xi s_\xi c_{\theta w_{1L}})/(s_\xi s_\xi s_{\theta w_{1L}}) \quad (20c)$$

$$\tan \psi'_{1R} = (c_\xi c_\xi - s_\xi s_\xi c_{\theta w_{1R}})/(s_\xi s_\xi s_{\theta w_{1L}}) \quad (20d)$$
For a spherical Earth and zero roll and zero pitch angles, the four semiscan angles are all equal:
\[ \theta_{\text{u0L}} = \theta_{\text{u1R}} = \theta_{\text{w0L}} = \theta_{\text{w1L}} = \theta_{\text{w0W}} \]  
and therefore, using Eqs. (20) and the angles \( \delta_i, \xi, \) and \( \theta_{\text{w0W}}, \) the instantaneous nominal azimuth angles of the four crossing points can be determined.

### III. Sensors in Pitch-Yaw Plane: Oblate Earth

**Noncircularity of the Horizon Caused by Oblateness**

In this study, Earth’s atmosphere and its radiance are not considered and therefore the oblate Earth’s radius given by Eq. (4.14) (Liù) is simplified to
\[ R = R_0(1 - f s^2 \lambda) \]  
where \( R_0 \) is the equatorial radius, 6378.14 km; \( R \) is the radius at latitude \( \lambda; s^2 = \sin^2 \lambda; \) and \( f \) is the ellipticity factor, \( f = 0.00335281, \) of the reference spheroid. It is noted that \( R \) in Eq. (22) is independent of the longitude \( \phi. \) Next, the angular radius \( \rho \) of the spheroid’s horizon given by Eq. (4.24) (Liu) at any subsatellite point is exceedingly complex, so it is simplified below by using \( R \) (Eq. (22)), by invoking binomial expansion in \( f, \) and then by ignoring all higher order terms of \( f, \) leading to
\[ \rho = \cot^{-1} \left[ \cot \rho_{0W} + \frac{1}{3} \cot \rho_{0L} \right] \]  
where \( s^2 = \cos^2 \lambda, \) and \( 2 \lambda = \sin 2 \lambda. \) Because \( f \) is very small, the angular radius \( \rho \) would be only slightly different from \( \rho_{0W} \) and therefore one may write
\[ \rho \approx \rho_{0W} + \varepsilon_{\rho} \]  
where \( |\varepsilon_{\rho}| \ll \rho_{0W}. \) Comparing Eq. (24) with Eq. (23) and using \( \varepsilon_{\rho} \ll 1 \) rad in trigonometric expansions yield
\[ \varepsilon_{\rho} = -f (c^2 \lambda \sin^3 \lambda \cos \rho_{0L}) + s^2 \lambda \tan \rho_{0L} + s 2 \lambda \sin 2 \lambda \]  
which shows that the noncircularity of the horizon is of the same order as \( f. \)

Another measure of oblateness is the slight difference between the semiscan angle \( \theta_{\text{w0L}} \) for a spherical Earth, with zero roll and zero pitch angles, and the four semiscan angles \( \theta_{\text{w0L}}, \) (\( i = 0, 1; j = L, R \)) for an oblate Earth (quasi because \( \theta_{\text{u0L}} \) and \( \theta_{\text{u1R}} \) may not be all equal but each is almost equal to \( \theta_{\text{w0L}}). \) Denoting this slight difference as \( \varepsilon_{\theta_{\text{w0L}}} \), we have
\[ \theta_{\text{w0L}} \approx \theta_{\text{w0W}} + \varepsilon_{\theta_{\text{w0L}}} \]  
where the small angle \( \varepsilon_{\theta_{\text{w0L}}} \) defined for an oblate Earth and zero roll and pitch angles should not be confused with the angle \( \Delta_w \) introduced earlier in Eq. (5) for nonzero roll and pitch angles. The angle \( \varepsilon_{\theta_{\text{w0L}}} \) is related to \( \varepsilon_{\theta_{\text{w0P}}} \) (\( \varepsilon_{\theta_{\text{w0P}}} \) at the crossing point \( ij \)) in a way established by generalizing Eq. (3) for an oblate Earth:
\[ c_{\theta ij} = c_{\theta ij} \]  
where \( c_{\theta ij} \) equals Earth’s angular radius at the crossing point \( ij. \) Differentiation of Eq. (27) furnishes
\[ \varepsilon_{\theta_{\text{w0L}}} = [s_{\theta ij} / (s_{\theta ij} s_{\theta_{\text{w0P}}})] \varepsilon_{\theta_{\text{w0P}}} \approx [s_{\theta ij} / (s_{\theta ij} s_{\theta_{\text{w0P}}})] \varepsilon_{\theta_{\text{w0L}}} \]  
where the first right-hand side (RHS) is exact and the second is based on the fact that \( e_{\theta_{\text{w0L}}} \) and \( e_{\theta_{\text{w0P}}} \) are both of the order 0.01 rad, so they both can be calculated using the spherical Earth parameters \( \rho_{0W} \) and \( \theta_{\text{w0W}} \) instead of the oblate Earth quantities \( \rho_{\theta} \) and \( \theta_{\text{w0W}} \). The quantities \( \varepsilon_{\theta_{\text{w0L}}} \) for the four crossing points are calculated using Eq. (25) where the azimuth \( \Psi \) corresponding to the crossing points are obtained from Eqs. (20), valid for a spherical Earth and nominal attitude of the spacecraft.

**Roll/Pitch Determination: Right Sensor**

For the oblate Earth, Eq. (4) corresponding to the roll measurement is generalized by substituting \( \rho_{\theta} \) (\( i = 0, 1 \)) for \( \rho_{0W} \) and \( \theta_{\text{w0W}} \) or \( \theta_{\text{w1L}} \) for \( \theta_{\text{w0L}}, \) to consider the in-crossing and out-crossing transit points. To linearize, we use the expansion (5), appending the subscripts \( 0R \) or \( 1R, \) and, additionally, use the expansion
\[ \rho_{\theta} = \rho_{0W} + \Delta_{\theta} \]  
where \( |\Delta_{\theta}| \ll \rho_{0W}. \) Comparing the expansions (24) and (29), we observe that, strictly speaking, \( \Delta_{\theta} \) corresponding to the nonzero roll/pitch angles will be different from \( \varepsilon_{\theta_{\text{w0L}}} \) corresponding to zero roll/pitch angles; but for small roll and pitch angles, this difference is negligible because \( \varepsilon_{\theta} \) varies slowly with the latitude \( \lambda \) and azimuth \( \Psi. \) Therefore, this difference will be ignored henceforth. Substitution of the expansions (5) and (29) in the generalized forms of Eq. (4) for the in-crossing and out-crossing points of the right sensor and linearization lead to
\[ \alpha_{1R} = \alpha_{1R} + \delta_{\alpha_{1R}} \]  
where the roll angle \( \alpha_{1R} \) measured by the right sensor is formally identical to Eq. (8):
\[ \alpha_{1R} = \frac{1}{2} K \left( \theta_{\text{u0L}} + \theta_{\text{u1R}} - 2 \theta_{\text{w0W}} \right) = \frac{1}{2} K \left( \Delta_{\theta} + \Delta_{\theta_{\text{w0L}}} \right) \]  
and the roll oblateness correction angle \( \delta_{\alpha_{1R}}, \) after using the relationship (28), is
\[ \delta_{\alpha_{1R}} = \frac{1}{2} K \left( e_{\theta_{\text{w0W}}} + e_{\theta_{\text{w1L}}} \right) \]  
Since the true roll angle \( \alpha_{1R} \) is defined relative to the geocentric vertical, Eqs. (30-32) indicate that \( \alpha_{1R} = 0 \) when \( \Delta_{\theta} = \Delta_{\theta_{\text{w0L}}} = 0, \) \( i = 0, 1), \) the measured roll angle \( \alpha_{1R} \) is then clearly nonzero. On the other hand, \( \alpha_{1R} = 0 \) whenever the geometric roll angle of the spacecraft equals \( \delta_{\alpha_{1R}}. \) Thus, one infers that the measured roll angle \( \alpha_{1R} \) must be a geodetic roll angle and
\[ \alpha_{1R} = \alpha_{1R} + \delta_{\alpha_{1R}} \]  
for geocentric and geodetic Earth-pointing orientations, respectively.

Consider the pitch angle \( \alpha_{2R} \) next. Equation (9) for the out-crossing point and the companion equation for the in-crossing point are generalized for an oblate Earth, as before. The linear analysis then leads to
\[ \alpha_{2R} = \alpha_{2R} + \delta_{\alpha_{2R}} \]  
where the measured pitch angle \( \alpha_{2R} \) using the right sensor, scanning counterclockwise, is
\[ \alpha_{2R} = \frac{1}{2} \cos \xi \left( \theta_{\text{u0L}} - \theta_{\text{u1R}} \right) = \frac{1}{2} \cos \xi \left( \Delta_{\theta} - \Delta_{\theta_{\text{w0L}}} \right) \]  
analogous to Eq. (10), and the corresponding pitch oblateness correction \( \delta_{\alpha_{2R}} \) is
\[ \delta_{\alpha_{2R}} = \frac{1}{2} \cos \xi \left( e_{\theta_{\text{w0W}}} - e_{\theta_{\text{w1L}}} \right) \]  
Arguing as before for the roll angle, \( \alpha_{2R} \) is a geodetic pitch angle measured by the right sensor and
\[ \alpha_{2R} = \alpha_{2R} + \delta_{\alpha_{2R}} \]  
for geocentric and geodetic Earth-pointing orientations, respectively.
Roll/Pitch Determination: Left Sensor

With the spacecraft at a roll angle $\alpha_L$, the tangency equation at the in-crossing and out-crossing points for the left sensor, scanning the oblate Earth, is

$$\theta_{OL} = \omega_{OL}(\xi - \alpha_L) + \Delta \phi_{OL} \alpha_L, \quad i = 0, 1 \quad (38)$$

analogous to Eq. (13a) for the right sensor. Substituting the expansions of $\omega_{OL}$ and $\Delta \phi_{OL}$ similar to the expansions (29) and (5), in Eq. (38) and performing a linear analysis again lead to

$$\alpha_1 = \alpha_{1mL} + \delta \alpha_1 \quad (39)$$

where the measured roll angle $\alpha_{1mL}$ and the roll oblateness correction $\delta \alpha_1$ are

$$\alpha_{1mL} = -\frac{1}{2} K (\theta_{OL} + \theta_{LL} - 2 \theta_{W9}) = -\frac{1}{2} K (\Delta \omega_{OL} + \Delta \omega_{LL}) \quad (40a)$$

$$\delta \alpha_{1L} = \frac{1}{2} K (\omega_{OL} + \omega_{LL}) \quad (40b)$$

$\alpha_{1mL}$ being formally the same as Eq. (14), except that $\alpha_{1mL}$ is a geodetic roll angle similar to $\alpha_{1mR}$ [Eq. (31)].

Regarding the pitch angle, we generalize Eq. (13b) and its companion for the left in-crossing point. The first-order expansion of the two equations then leads to

$$\alpha_2 = \alpha_{2mL} + \delta \alpha_2 \quad (41)$$

where the measured geodetic pitch angle and the associated oblateness correction are, for counterclockwise scanning,

$$\alpha_{2mL} = \frac{1}{2} \cos \xi (\theta_{OL} - \theta_{LL}) = \frac{1}{2} \cos \xi (\Delta \omega_{OL} - \Delta \omega_{LL}) \quad (42a)$$

$$\delta \alpha_{2L} = \frac{1}{2} \cos \xi (\omega_{OL} - \omega_{LL}) \quad (42b)$$

The right-hand sides of Eqs. (15a) and (42a) both for ccw scan are formally the same, although $\theta_{W9}$ and $\theta_{LL}$ in Eq. (42a) pertain to scanning an oblate Earth and those in Eq. (15a) pertain to scanning a spherical Earth.

Roll/Pitch Determination: Both Sensors

When a spacecraft is fitted with both right and left sensors, the average geocentric roll angle is obtained by adding Eqs. (30) and (39), leading to

$$\alpha_1 = \alpha_{1m} + \delta \alpha_1 \quad (43)$$

where the measured geodetic roll angle $\alpha_{1m}$ is formally the same as Eq. (16) and the roll oblateness correction $\delta \alpha_1$ is

$$\delta \alpha_1 = \frac{1}{2} K (\omega_{OL} + \omega_{LL} - \omega_{W9} - \omega_{W9}) \quad (44)$$

regardless of the scan direction. The pitch angle is likewise obtained by adding and averaging Eqs. (34) and (41), arriving at

$$\alpha_2 = \alpha_{2m} + \delta \alpha_2 \quad (45)$$

where $\alpha_{2m}$ is formally the same as the right-hand sides of Eqs. (17) and the pitch oblateness correction $\delta \alpha_2$ is

$$\delta \alpha_2 = \frac{1}{2} \cos \xi (\omega_{W9} - \omega_{W9} + \omega_{LL} - \omega_{W9}) \quad (46a)$$

for counterclockwise scanning and

$$\delta \alpha_2 = \frac{1}{2} \cos \xi (\omega_{W9} - \omega_{W9} + \omega_{LL} - \omega_{W9}) \quad (46b)$$

for clockwise scanning. While the pitch oblateness corrections (36) and (42b) resemble Eq. (9) of Ref. 9 (wherein the sensor scans clockwise), the roll oblateness corrections above cannot be related so easily with Eq. (5) of Ref. 9 because the partial derivatives therein are not worked out.

Illustration

The sensor tilt angle $\xi$ shown in Fig. 1 is introduced so as to increase the scanwidth $2\theta_9$ of Earth and the slope $K$, increasing in turn the measurability of the roll and pitch angles. For a spacecraft in a circular orbit, Fig. 4 illustrates the semiscan angle $\theta_{OL}$ vs a spherical Earth vs the tilt angle $\xi$ for three semicone angles $\delta_0$. As expected, $\theta_{OL}$ increases with both $\delta_0$ and $\xi$, although, as $\delta_0$ increases, the influence of $\xi$ on $\theta_{OL}$ weakens. The corresponding variation in the slope $K$ is illustrated in Fig. 5. A tempting conclusion from Figs. 4 and 5 is to select $\delta_0 = 60$ deg and $\xi = 20$ deg, but further parametric studies are desired to arrive at an optimum selection of these angles.

The roll oblateness corrections associated with the right and the left sensors, separately and combined [Eqs. (32), (40b), and (44)], are illustrated in Fig. 6 starting from the ascending node. The correction $-\delta \alpha_{1L}$, instead of $\delta \alpha_{1L}$, is shown to aid its visual comparison with $\delta \alpha_{1L}$. Over one orbital period, the magnitudes of the corrections associated with the right and the left sensors interchange. Surprisingly, the maximum roll correction for one sensor for a 100-min near-polar orbit can be as much as 0.43 deg, but if the two sensors are used together, this amplitude reduces to 0.09 deg. These results are in accord with those in Fig. 7 of Ref. 6, except that, apparently, Alex and Shrivastava have plotted negative of the correction. The average roll correction $\delta \alpha_1$ is zero at the equator ($\cos \gamma = 0$, $\pi$) and maximum near poles ($\cos \gamma = \pm \pi / 2$), but instead of varying at twice the orbit rate, as stated in Ref. 11, it varies sinusoidally at the orbit rate. The significance of the quantity $-\cos \gamma$ in Fig. 6 is explained in Sec. IV.

The pitch oblateness corrections for the same parameters as those in Fig. 6 are illustrated in Fig. 7. Unlike the roll corrections, the
positive \( \lambda \), so we introduce the right-handed spherical coordinates \( r, \lambda, \phi \) (in this sequence), where \( \lambda_c = 90 \deg - \lambda \). The associated dextral unit vector triad is \( \mathbf{u}_r, \mathbf{u}_\phi, \mathbf{u}_\phi \), where \( \mathbf{u}_r \) is a radial unit vector, \( \mathbf{u}_\phi \) is the unit vector in the direction of increasing \( \lambda_c \), and \( \mathbf{u}_\phi \) is the unit vector along the local east (\( \mathbf{u}_\phi = \hat{E} \)). The outward normal \( \mathbf{n} \) is then given by

\[
\mathbf{n} = \mathbf{u}_r - \mathbf{f} \sin 2\lambda_c \mathbf{u}_\phi
\]

where \( \mathbf{f}^2 \) and higher order terms have been ignored. Clearly, \( \mathbf{n} \) is in the plane formed by \( \mathbf{u}_r \) and \( \mathbf{u}_\phi \) (the plane of Fig. 8), and therefore, the deviation angle \( \eta \) between \( \mathbf{n} \) and \( \mathbf{u}_r \) is about the unit vector \( \mathbf{u}_\phi \) passing through the spacecraft mass center at a height \( h \) and the geocentric unit vector \( \mathbf{u}_r = -\mathbf{c}_3 \) at the spacecraft mass center is

\[
\eta \approx \mathbf{f} \mathbf{R}_{eq}/(\mathbf{R}_{eq} + h)
\]

about the east unit vector at the spacecraft location and positive, as shown in Fig. 8.

### Roll/Pitch Components of Deviation

We note from Fig. 3 that

\[
\mathbf{E} = \cos \gamma \mathbf{e}_1 + \sin \gamma \mathbf{e}_2
\]

Hence, the small-angle vector \( \eta \mathbf{E} \) has the components \( \eta \mathbf{c}_1 \) and \( \eta \mathbf{c}_2 \) about the \( \mathbf{c}_1 \) and \( \mathbf{c}_2 \) vectors, respectively. To render explicit the dependence of these components on the orbit angle \( \omega_{0ft} \), define

\[
\eta_{max} \equiv \mathbf{f} \mathbf{R}_{eq}/(\mathbf{R}_{eq} + h)
\]

Reference 13 then shows that the components of \( \eta \mathbf{E} \) are

\[
\eta \cos \gamma = \eta_{max} \sin 2\iota \sin \omega_{0ft} \quad \eta \sin \gamma = \eta_{max} \sin^2 \iota \sin 2\omega_{0ft}
\]

and the angle \( \eta \) varies with the orbit angle as follows:

\[
\eta = 2\eta_{max} \sin \iota \sin \omega_{0ft} \left(1 - \sin^2 \iota \sin^2 \omega_{0ft}\right)^{1/2}
\]

To relate the roll and pitch components [Eqs. (51)] with the oblateness corrections \( \delta_{a1} \) and \( \delta_{a2} \), recall that the geodetic roll and pitch angles are zero (\( \alpha_{a1} = 0 = \alpha_{a2} \)) when the spacecraft's yaw axis \( \mathbf{b}_3 \) passes through the geodetic subsatellite point \( p \), that is, when \( \mathbf{b}_3 = -\mathbf{n} \). This implies that, to align \( \mathbf{b}_3 \) with the geodetic nadir \(-\mathbf{n}, \mathbf{c}_3 \)
the spacecraft must turn by a roll angle $\alpha_1$ and a pitch angle $\alpha_2$ equal to
\[ \alpha_1 = -\eta \cos \gamma \quad \alpha_2 = -\eta \sin \gamma \]  
(53)

the negative signs stemming from $c_1 \times b_1 = -nE = \alpha_1 c_1 + \alpha_2 c_2$. These quantities were illustrated in Figs. 6 and 7, wherein we now observe that they agree very well with the oblateness corrections formulated in Sec. III for a pair of horizon sensors. Therefore, when there are two sensors, the oblateness corrections are
\[ \delta \alpha_1 \approx -\eta \cos \gamma \quad \delta \alpha_2 \approx -\eta \sin \gamma \]  
(54)

which is a handy conclusion, for the corrections then need not be calculated using the analysis of Sec. III. Indeed, substituting $\delta \alpha_1$ and $\delta \alpha_2$ from Eq. (54) in Eqs. (43) and (45) and recalling Eqs. (48) and (50), we arrive at
\[ \alpha_1 = \alpha_{1m} - \eta_{mx} \cos \gamma \sin 2\lambda \]  
(55a)
\[ \alpha_2 = \alpha_{2m} - \eta_{mx} \sin \gamma \sin 2\lambda \]  
(55b)

which are the same as those stated without proof by Lebsock and Eterno\textsuperscript{11} (1987 version). Perhaps more importantly, we also conclude that if only one sensor is onboard the spacecraft, Eqs. (54) and (55) become invalid and the corrections then must be calculated as shown in Sec. III.

V. Aft and Forward Horizon Sensors in Roll-Yaw Plane

While packaging various instruments in a spacecraft bus, it is sometimes not feasible to install the scanning horizon sensors in the pitch-yaw plane; for example, the solar arrays might be occluding the sensor’s field of view. In that event, the Earth sensors may be mounted on the aft and/or forward face of the spacecraft. Moreover, if both aft and forward sensors are used, they may scan Earth in the same direction or in the opposite direction. In the latter case, the net spin angular momentum of the two sensors is virtually zero, although the angular momentum of a scanning sensor is usually very small to begin with. In any event, Fig. 9 depicts the scanpaths of aft and forward sensors, tilted in the roll-yaw plane and each spinning anticlockwise about their own scan axes. Below we develop the roll and pitch angles as measured by these sensors, individually and together, and the associated oblateness corrections.

Aft Sensor

Equation (3), which determines the semiscan angle $\theta_{a0}$ of the sensor for a spherical Earth with the spacecraft in the zero roll/pitch orientation, still applies. Developing the math model and performing the linear analysis in the same fashion as in Secs. II and III, we arrive at the geocentric roll angle equation similar to Eq. (30), with the subscript $R$ replaced by the subscript $A$ (for aft). The geodetic measured roll angle $\alpha_{1m,A}$ and the associated oblateness correction $\delta \alpha_{1A}$ in terms of the measured scan angles are now
\[ \alpha_{1m,A} = \frac{1}{2} \cos \xi (\theta_{a1A} - \theta_{a0A}) \]  
(56a)
\[ \delta \alpha_{1A} = \frac{1}{2} \cos \xi (e_{a0A} - e_{u1A}) \]  
(56b)

Equation (56a) is clearly right because, for a positive roll angle about $e_1$, the in-crossing scan angle $\theta_{a1A}$ up to the reference mark will be smaller than the out-crossing scan angle $\theta_{a0A}$. However, notice that the form of Eqs. (56) is similar to the measured pitch angle $e_{2m,R}$ [Eq. (35)] and the associated oblateness correction $\delta \alpha_{2R}$ [Eq. (36)], because, earlier, the scan axis was in the pitch-yaw plane and now it is in the roll-yaw plane. The geocentric pitch angle, on the other hand, is given by an equation similar to Eq. (34), with the subscript $R$ replaced by the subscript $A$. The measured geodetic pitch angle $\alpha_{2m,A}$ and the associated oblateness correction $\delta \alpha_{2A}$ are defined by
\[ \alpha_{2m,A} = \frac{1}{2} K (\theta_{a0A} + \theta_{u1A} - 2\theta_{a0}) \]  
(57a)
\[ \delta \alpha_{2A} = -\frac{1}{2} K (e_{a0A} + e_{u1A}) \]  
(57b)

which are analogous to, as expected, the roll equations (31) and (32).

Forward Sensor

This sensor spins counterclockwise about an axis tilted by an angle $\xi$ from the unit vector $e_1$ in the roll-yaw plane. The corresponding scanpath on the Earth disk is shown in Fig. 9. The roll and pitch angles are given by equations similar to those for the aft sensor, with the subscript $A$ replaced by $F$ (for forward). The measured angles and the oblateness corrections are the following:
\[ \alpha_{1m,F} = \frac{1}{2} \cos \xi (\theta_{a1F} - \theta_{a0F}) \]  
(58a)
\[ \delta \alpha_{1F} = \frac{1}{2} \cos \xi (e_{a0F} - e_{u1F}) \]  
(58b)
\[ \alpha_{2m,F} = -\frac{1}{2} K (\theta_{a0F} + \theta_{u1F} - 2\theta_{a0}) \]  
(59a)
\[ \delta \alpha_{2F} = \frac{1}{2} K (e_{a0F} + e_{u1F}) \]  
(59b)

Because the aft sensor and the forward sensor are spinning oppositely, Eqs. (58) and (59) appear negative of Eqs. (56) and (57). Also, as in the case of aft sensor, the roll equations (58) resemble the pitch equations (35) and (36) and the pitch equations (59) resemble the roll equations (31) and (32).

Aft and Forward Sensors

When the roll and pitch measurements from the two sensors and the associated oblateness corrections are added and averaged, Eqs. (43) and (45) follow where, now,
\[ \alpha_{1m} = \frac{1}{2} \cos \xi (\theta_{a1A} + \theta_{a0A} + \theta_{a1F} - \theta_{a0F}) \]  
(60a)
\[ \delta \alpha_1 = \frac{1}{2} \cos \xi (e_{a0F} - e_{u1F} - e_{a0A} - e_{u1A}) \]  
(60b)
\[ \alpha_{2m} = \frac{1}{2} K (\theta_{a0A} + \theta_{u1A} - \theta_{a1F} - \theta_{a0F}) \]  
(61a)
\[ \delta \alpha_2 = \frac{1}{2} K (e_{a0F} + e_{u1F} - e_{a0A} - e_{u1A}) \]  
(61b)

The evaluation of the oblateness corrections $\delta \alpha_1$ and $\delta \alpha_2$ requires determination of $e_{u,ij}$ ($i = 0, 1; j = A, F$), which are governed by Eqs. (25) and (28). The horizon crossing points $\psi_{ij}$ appearing in $e_{u,ij}$ are determined as follows.

Horizon Crossing Points

Calling upon the definition (19) of the crossing point $\psi_{ij}$ ($i = 0, 1; j = A, F$) and determining the components along $e_1$ and $e_2$ of the four optic vectors, for zero roll/pitch attitude, we arrive at the following:

Aft out-crossing
\[ \tan \psi_{1A} = -s_1 c_{a1A}/(c_0 c_{e1} + s_1 s_{e1} c_{a0A}) \]  
(62a)
This general agreement in roll, however, does not carry over to the pitch axis, as we see in Fig. 11, wherein the individual pitch corrections are nearly four times larger than the average pitch correction, which in turn is nearly the same as the closed-form pitch correction \((-\eta \sin \gamma)\). Figures 10 and 11, just as Figs. 6 and 7, suggest that if both aft and forward sensors are employed, the closed-form expressions \(-\eta \cos \gamma\) and \(-\eta \sin \gamma\) can be used for oblateness correction; if not, the corrections may be computed on the ground for the selected single sensor and stored in the flight computer in a table lookup form before flight.

VI. Roll/Pitch Altitude Corrections

Simple, linear, attitude determination equations of the preceding allow for the evaluation of roll/pitch corrections to compensate for the spacecraft altitude variations caused by eccentricity of the orbit. Because both the oblateness corrections and the altitude corrections are small, they can be treated separately and so, in this section, we focus on the altitude corrections.

The first quantity affected by the spacecraft altitude variation \(\delta h\) is the angular radius \(\rho_0\) of the Earth disk. Differentiating Eq. (1), we obtain

\[
\delta \rho_0 = -\left(\tan \theta_{\omega_0}\right) \delta h / (R_{\omega_0} + h)
\]

where \(\delta \rho_0\) is the variation in \(\rho_0\) caused by \(\delta h\). This, then, changes the scanwidth \(2\theta_{\omega_0}\) of Earth; quantitatively, Eq. (3) yields

\[
\delta \theta_{\omega_0} = -\sin \theta_{\omega_0} \tan \rho_0 \delta h / (R_{\omega_0} + h) \sin \delta_{\omega} \cos \xi \sin \theta_{\omega_0}
\]

Examining Eqs. (6) and (7) we observe that when a single sensor is used for attitude determination, the variation \(\delta \theta_{\omega_0}\) affects the attitude measurement accuracy directly as well as through the change in the slope \(K\). The change \(\delta K\) is determined by differentiating Eq. (7):

\[
\delta K = \delta \theta_{\omega_0} (\cot \delta_{\omega} \cos \theta_{\omega_0} - \tan \xi) / (\cot \delta_{\omega} - \tan \xi \cos \theta_{\omega_0})^2
\]

The altitude corrections for individual sensors can in turn be formulated as shown below. However, we note that if the instantaneous altitude of the spacecraft is known and used in flight to update the slope \(K\) and the semiscan width \(\theta_{\omega_0}\), no additional altitude correction will be required because, then, the measured scan angle \(2\theta_{\omega_0}\), \(K\), and \(\theta_{\omega_0}\) all pertain to the same altitude. However, the following perturbation analysis helps in the predetermination of the roll/pitch corrections for a possible range of the altitude variation \(\delta h\).

Left and Right Sensors

Inspecting the roll equation (6) and the pitch equation (10), we observe that the pitch angle requires no altitude correction because from Fig. 11, wherein the individual pitch corrections are nearly four times larger than the average pitch correction, which in turn is nearly the same as the closed-form pitch correction \((-\eta \sin \gamma)\). Figures 10 and 11, just as Figs. 6 and 7, suggest that if both aft and forward sensors are employed, the closed-form expressions \(-\eta \cos \gamma\) and \(-\eta \sin \gamma\) can be used for oblateness correction; if not, the corrections may be computed on the ground for the selected single sensor and stored in the flight computer in a table lookup form before flight.

This general agreement in roll, however, does not carry over to the pitch axis, as we see in Fig. 11, wherein the individual pitch corrections are nearly four times larger than the average pitch correction, which in turn is nearly the same as the closed-form pitch correction \((-\eta \sin \gamma)\). Figures 10 and 11, just as Figs. 6 and 7, suggest that if both aft and forward sensors are employed, the closed-form expressions \(-\eta \cos \gamma\) and \(-\eta \sin \gamma\) can be used for oblateness correction; if not, the corrections may be computed on the ground for the selected single sensor and stored in the flight computer in a table lookup form before flight.
Fig. 12 Pitch altitude corrections for aft sensor vs altitude variation with measured pitch angle equal to 2 deg.

Aft and Forward Sensors

Examining the roll and pitch measurement equations (56a) and (57a) or (58a) and (59a), we infer that, unlike the case of the left and right sensors, the roll measurements now make no altitude correction but the pitch measurements do. Differentiating the pitch measurement equations (57a) and (59a) corresponding to the aft and forward sensors, the altitude corrections associated with each sensor separately are found to be

\[ \delta \alpha_{2A} = (\delta K/K) \alpha_{2m,A} - K \delta \theta_{w\Phi} \]  
\[ \delta \alpha_{2F} = (\delta K/K) \alpha_{2m,F} + K \delta \theta_{w\Phi} \]  

When both sensors are used together, the combined altitude correction is

\[ \delta \alpha_{2} = (\delta K/K) \alpha_{2m} \]  

where the measured pitch angle \( \alpha_{2m} \) is given by Eq. (62a).

Illustration

Figure 12 depicts the pitch corrections for an aft sensor when the nominal altitude varies by ±50 km and the other nominal parameters are fixed as noted in the figure. The bias correction \(- K \delta \theta_{w\Phi}\) predominates over the slope correction. For example, when \( \delta h = 50 \) km, \(- K \delta \theta_{w\Phi} \approx 0.95 \) deg. Compare this with the pitch correction \((\delta K/K) \alpha_{2m,A}\) caused by the slope change \(\delta K\), equal to \(-0.04\) deg when the measured pitch angle is 2 deg. If the true altitude is known, through navigation or ephemeris or radar altimeter, the pitch altitude correction will not be needed altogether; if not, a forward sensor can be employed whose pitch measurements, combined with the aft sensor measurements, will eliminate the bias error \(K \delta \theta_{w\Phi}\). A small residual error \((\delta K/K) \alpha_{2m}\) will still remain, but this may be within the error budget.

VII. Concluding Remarks

Roll/pitch oblateness corrections to the scanning horizon sensor data can be determined in two ways: 1) by decomposing the angular deviation of the geocentric normal from the geodetic normal into the roll and pitch components. The two approaches yield essentially the same corrections if the Earth-pointing spacecraft is equipped with multiple scanning sensors—left and right along the pitch axis or aft and forward along the roll axis or, by the same token, four diagonal sensors. In that event, the second approach, which furnishes closed-form expressions for the roll/pitch oblateness corrections, can be used onboard and is highly recommended. However, if a spacecraft is outfitted with only one sensor, the corrections about one axis—roll if the sensor is along the pitch axis and pitch if the sensor is along the roll axis—is far different from that predicted by the geocentric/geodetic angular deviation approach. The corrections then may be calculated on the ground for the desired sensor location and stored in the flight computer using the crossing point approach. The paper treats two most common arrangements of the horizon sensors—a pair tilted in the pitch–yaw plane and one in the roll–yaw plane—and obtains, through linear perturbation analysis, the corrections to compensate Earth’s oblateness and the spacecraft’s attitude variations arising from the orbit eccentricity.

References