Wavelet based spectral finite element for analysis of coupled wave propagation in higher order composite beams

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Abstract

In this paper, a spectrally formulated wavelet finite element is developed and is used to study coupled wave propagation in higher order composite beams. The beam element has four degrees of freedom at each node, namely axial and transverse displacements, shear and contraction. The formulation is used to perform both frequency and time domain analysis. The formulation is similar to conventional FFT based Spectral Finite Element (FSFE) except that, here Daubechies wavelet basis is used for approximation in time to reduce the governing PDE to a set of ODEs. The localized nature of the compactly supported Daubechies wavelet basis helps to circumvent several problems associated with FSFE due to the required assumption of periodicity, particularly for time domain analysis. However, in Wavelet based Spectral Finite Element (WSFE), a constraint on the time sampling rate has to be placed to avoid the introduction of spurious dispersion in the analysis. Numerical experiments are performed to study spectrum and dispersion relation. In addition, the wave propagation in finite length structures due to broad band impulse loading is studied to bring out the higher order effects. Simultaneous existence of various propagating modes are graphically captured using modulated sinusoidal pulse excitation. In all the cases comparison are provided with FSFE.

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1. Introduction

Wavelets are recently being widely used for numerical solutions of partial differential equations (PDEs) [1–5]. Reference [6] provides a review of such wavelet based techniques. The orthogonal, compactly supported wavelet basis of Daubechies [7,8] can provide accurate and stable representation of differential operations even in region of strong gradients or oscillations. In addition, the orthogonal wavelet basis have the inherent advantage of multi-resolution analysis over the traditional methods.

Numerical solution of elastic wave equations require high accuracy in numerical differentiation and at the same time have larger spatial grids and time steps to make it computationally efficient. Wave propagation problems deal with loading that have very high frequency content and Finite Element (FE) formulation for such problems require large system size to capture all the higher modes. These problems are usually solved in frequency domain using Fourier methods, which can in principle achieve high accuracy in numerical differentiation. One such method is FFT based Spectral Finite Element Method (FSFEM) [9].

In FSFEM, first the governing PDEs are transformed to ODEs in spatial dimension using FFT in time. These ODEs are then usually solved exactly, which are used as interpolating functions for FSFE formulation. This results in exact mass distribution and hence, in absence of any discontinuity, one element is sufficient to handle 1-D structure of any length and this reduces the system size substantially. Such FSFE for elementary rod
The main drawback of Fourier based spectral approach is that it cannot handle waveguides of short lengths. This is because, short length forces multiple reflections at smaller time scales. Since Fourier transforms are associated with a finite time window, shorter lengths of waveguide do not allow the response to die down within the chosen time window, irrespective of the type of damping used in modeling. This forces the response to wrap around, that is the remaining part of the response beyond the chosen time window, will start appearing first. This totally distorts the response. It is in such cases compactly supported wavelets, which have localized basis functions can be efficiently used for waveguides of short lengths. Different wavelet based modeling techniques for simulation of wave propagation have been presented in [15–17].

WSFE formulation [18] is very similar to FSFEM, except that Daubechies scaling functions are used for approximation in time. This reduces the PDEs to coupled ODEs which are decoupled using eigenvalue analysis. The decoupled ODEs are then solved similarly as in FSFEM. The wavelet analysis can be adapted to a finite domain and initial values can be imposed using the wavelet extrapolation technique [19–21]. This removes the problem associated with “wrap around” due to the assumed periodicity of solutions in FSFEM and thus result in smaller time window for the same problem. For similar reason, WSFE can be used for analysis of undamped structures where FSFEM does not work. However, WSFE can also be formulated by considering periodic boundary condition and for this case the results are expected to be similar to those obtained using FSFEM.

Though FSFEM encounters several problems in time domain analysis of wave propagation, it is the only method used to study the various frequency dependent characteristics of waves namely the spectrum and dispersion relation. In FSFEM the elastic wave equations are transformed to frequency domain using FFT. Thus the spectrum (frequency dependence of wavenumber) and dispersion (frequency dependence of wave speeds) relations can be obtained directly from analysis in the transformed domain. A correspondence can be established between the transformed ODEs in periodic WSFEM with those obtained in FSFEM [22]. This study helps in using the formulated WSFE directly for such frequency domain analysis of wave propagation and also a-priori determination of sampling rate to avoid the introduction of spurious modes in the analysis. This sampling rate depends on the excitation frequency and order of basis function used.

Wave propagation in composite structures is an important research area since such structures are very often subjected to high velocity impact loading like tools drop, gust, etc. Wave propagation analysis is also extensively used for structural health monitoring studies. Recently, wavelet analysis of wave propagation for damage detection is being reported in literature [23]. In [24], wavelet analysis of the response to impulse loading is done to detect delamination and impact damages in composite beam and plates. Delamination detection in composite structures is done in [25] by wavelet based post processing of response due to input wave. Wavelet based wave propagation analysis is used for determining source location in composite laminate in [26].

In this paper, WSFE is formulated for a higher order composite beam to study coupled wave propagation. The formulated element has four degrees of freedom, namely axial and transverse displacements, shear and contraction at each node. The dynamics of such higher order beam structures subjected to high frequency loading or impact introduces certain effects which are absent in the elementary counterpart. Wave propagation studies in a deep metallic beam show that the flexural loading causes an additional propagating shear mode due to shear deformation and rotary inertia [13]. In addition, in composites due high ratios of elastic moduli, the effects of transverse shear deformation are even more prominent. When the same deep metallic member is subjected to axial load, an additional propagating mode due to Poisson’s contraction appears [14]. The Poisson’s contraction in metallic members of circular cross-section was first introduced by Mindlin–Herrmann [27] to study the wave propagation effects in deep rods. Hence, the formulation with Poisson’s or lateral contraction effects are normally referred to as Mindlin–Herrmann formulation. This theory was later extended to rectangular cross-section in Ref. [14]. It is shown in [14] that, the introduction of Poisson’s or lateral contraction in model makes the axial wave dispersive and the degree of dispersions increases with the increase in depth. Thus higher order beam model is essential for accurate analysis, particularly for composite beams with strong asymmetry. The wave propagation behavior in such asymmetric composite beam is much more complex due the axial and flexural coupling. FSFE has been developed for such higher order composite beam by Mahapatra and Gopalakrishnan [28]. FSFE is also developed to study axial-flexural coupled wave propagation in asymmetric composite beam in [29]. Apart from this, FSFE formulation for wave propagation analysis in functionally graded beam is presented in [30].

The paper is organized as follows. In Section 2, a brief overview of the orthonormal bases of compactly supported wavelets are presented. In next sections, the details of wavelet based spectral element formulation is given for laminated composite beam with axial, transverse, shear and contractional degree of freedom at each node. Following these sections, a section on numerical
experiments is presented. In this section, first, WSFEM is
used to obtain the frequency dependent parameters, i.e. wavenumber and wave speeds for the higher order
composite beam. Next, the formulated element is used
to study axial-flexural-shear-contraction coupled wave
propagation in finite length beam due to broad-band im-
pulse loading. Non-periodic WSFEM is used for such
simulations. Finally, simultaneous existence of various
coupled propagating modes are studied using modulated
sinusoidal loading applied to an infinite beam. For such
infinite structures, periodic WSFEM is used. In most of
the above cases, comparison is provided either with con-
tventional FE or FSFEM results to highlight the different
advantages and limitations of the developed WSFE. The
paper ends with some important conclusions.

2. Daubechies compactly supported wavelets

In this section, a concise review of orthogonal basis of
Daubechies wavelets [7,8] is provided. Wavelets, \( \psi_{j,k}(t) \)
forms compactly supported orthonormal basis for \( L^2(\mathbb{R}) \). The wavelets and associated scaling functions
\( \phi_{j,k}(t) \) are obtained by translation and dilation of single
functions \( \psi(t) \) and \( \phi(t) \), respectively.

\[
\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k), \quad j, k \in \mathbb{Z}
\]

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\]

The scaling functions \( \phi(t) \) are derived from the dilation or scaling equation,

\[
\phi(t) = \sum_k a_k \phi(2t - k)
\]

and the wavelet function \( \psi(t) \) is obtained as

\[
\psi(t) = \sum_k (-1)^k a_{1-k} \phi(2t - k)
\]

\( a_k \) are the filter coefficients and they are fixed for specific
wavelet or scaling function basis. For compactly supported

wavelets only a finite number of \( a_k \) are nonzero.

The filter coefficients \( a_k \) are derived by imposing cer-
tain constraints on the scaling functions which are as follows:

1. The area under scaling function is normalized to
   one;
   \[
   \int_{-\infty}^{\infty} \phi(t) dt = 1
   \]

2. The scaling function \( \phi(t) \) and its translates are
   orthonormal;
   \[
   \int_{-\infty}^{\infty} \phi(t) \phi(t + k) dt = \delta_{0,k}, \quad k \in \mathbb{Z}
   \]

(3) wavelet function \( \psi(t) \) has \( M \) vanishing moments

\[
\int_{-\infty}^{\infty} \psi(t)^m dt = 0, \quad m = 0, \ldots, M
\]

The number of vanishing moments \( M \) denotes the order
\( N \) of the Daubechies wavelet, where \( N = 2M \).

The translates of the scaling and wavelet functions on
each fixed scale \( j \) form orthogonal subspaces,

\[
V_j = \{ 2^j \phi(2^j t - k) ; j \in \mathbb{Z} \}
\]

\[
W_j = \{ 2^j \psi(2^j t - k) ; j \in \mathbb{Z} \}
\]

such that \( V_j \) forms a sequences of embedded subspaces
\( \{0\}, \ldots, \subset V_{-1}, \subset V_0, \subset V_1, \ldots, \subset L^2(\mathbb{R}) \)
and

\[
V_{j+1} = V_j \oplus W_j
\]

Let \( P_j(f) \) be approximation of a function \( f(t) \) in \( L^2(\mathbb{R}) \)
using \( \phi_{j,k}(t) \) as basis, at a certain level (resolution) \( j \), then

\[
P_j(f)(t) = \sum_k c_{j,k} \phi_{j,k}(t), \quad k \in \mathbb{Z}
\]

where \( c_{j,k} \) are the approximation coefficients. Let \( Q_j(f)(t) \)
be the approximation of the function using \( \psi_{j,k}(t) \) as
basis, at the same level \( j \).

\[
Q_j(f)(t) = \sum_k d_{j,k} \psi_{j,k}(t), \quad k \in \mathbb{Z}
\]

where \( d_{j,k} \) are the detail coefficients. The approxima-
tion \( P_{j+1}(f)(t) \) to the next finer level of resolution \( j + 1 \)
is given by

\[
P_{j+1}(f)(t) = P_j(f)(t) + Q_j(f)(t)
\]

This forms the basis of multi-resolution analysis associ-
ated with wavelet approximation.

3. Reduction of wave equations to ODEs

The governing differential wave equations for higher
order composite beam derived in [28] are

\[
I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^2 \phi}{\partial t^2} - A_{11} \frac{\partial^2 u}{\partial x^2} + B_{11} \frac{\partial^2 \phi}{\partial x^2} - A_{13} \frac{\partial \psi}{\partial x} = 0
\]

\[
I_2 \frac{\partial^2 \psi}{\partial t^2} + I_3 \frac{\partial^2 \psi}{\partial t^2} + A_{13} \frac{\partial u}{\partial x} - B_{13} \frac{\partial \phi}{\partial x} + A_{33} \psi
\]

\[
- B_{55} \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) - D_{55} \frac{\partial^2 \psi}{\partial x^2} = 0
\]

\[
I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^2 \psi}{\partial t^2} - A_{55} \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \phi}{\partial x} \right) - B_{55} \frac{\partial^2 \psi}{\partial x^2} = 0
\]
\[
\frac{I_2}{I_1} \frac{\partial^2 \phi}{\partial t^2} - I_1 \frac{\partial^2 u}{\partial t^2} - A_{ss} \left( \frac{\partial \psi}{\partial x} - \phi \right) - B_{ss} \frac{\partial \psi}{\partial x} + B_{11} \frac{\partial^2 u}{\partial x^2} - D_{11} \frac{\partial^2 \phi}{\partial x^2} + B_{11} \frac{\partial \psi}{\partial x} = 0
\]

(18)

where \( u(x,t), w(x,t), \phi(x,t) \) and \( \psi(x,t) \) are the axial, flexural, shear and contractional displacements, respectively (see Fig. 1(a)). The stiffness coefficients are functions of ply properties, orientations, etc. and are obtained by integrating over the beam cross-section as

\[
[A_{ij}, B_{ij}, D_{ij}] = \int_{z_i}^{z_{i+1}} \bar{Q}_{ij}[1, z, z^2] b \, dz
\]

(19)

where \( \bar{Q}_{ij} \) are the transformed stiffness coefficients of each ply. \( z_i \) and \( z_{i+1} \) are the thickness coordinate of the \( i \)th layer and \( b \) is the corresponding width. Similarly, the inertial constants are obtained as

\[
[I_0, I_1, I_2] = \int_{z_i}^{z_{i+1}} \rho[1, z, z^2] b \, dz
\]

(20)

where \( \rho \) is the mass density. The displacement fields and details of the derivation of the above equations (Eqs. (15)–(18)) are given in [28].

The first step of formulation of WSFE is the reduction of the governing differential wave equations (Eqs. (15)–(18)) to ODEs using Daubechies scaling functions for approximation in time. Let \( u(x,t) \) be discretized at \( n \) points in the time window \([0, T_f] \). Let \( \tau = 0, 1, \ldots, n - 1 \) be the sampling points, then

Fig. 1. (a) Beam cross-section and the displacements; (b) Composite beam element with nodal displacements and forces; (c) An infinite beam to observe non-dispersively propagating modes.
where \( \Delta t \) is the time interval between two sampling points. The function \( u(x, t) \) can be approximated by scaling function \( \phi(t) \) at an arbitrary scale as

\[
u(x, t) = u(x, t) = \sum_k u_k(x) \phi(t - k), \quad k \in \mathbb{Z}
\]

(22)

where \( u_k(x) \) (referred as \( u_k \) hereafter) are the approximation coefficient at a certain spatial dimension \( x \). The other displacements \( w(x, t), \phi(x, t), \psi(x, t) \) can be transformed similarly and Eq. (15) can be written as

\[
\frac{I_0}{\Delta t^2} \sum_k u_k \phi''(t - k) - \frac{I_1}{\Delta t^2} \sum_k \phi_k \phi''(t - k) + \sum_k \left( -A_{11} \frac{d^2 u_k}{dx^2} + B_{11} \frac{d^2 \phi_k}{dx^2} - A_{13} \frac{d \psi_k}{dx} \right) \phi(t - k) = 0
\]

(23)

Taking inner product on both sides of Eq. (23) with \( \phi(t - j) \), where \( j = 0, 1, \ldots, n - 1 \) we get,

\[
\frac{I_0}{\Delta t^2} \sum_k u_j \int \phi''(t - k) \phi(t - j) dt - \frac{I_1}{\Delta t^2} \sum_k \phi_k \phi''(t - k) \phi(t - j) + \sum_k \left( -A_{11} \frac{d^2 u_k}{dx^2} + B_{11} \frac{d^2 \phi_k}{dx^2} - A_{13} \frac{d \psi_k}{dx} \right) \phi(t - k) \phi(t - j) = 0
\]

(24)

The translates of scaling functions are orthogonal, i.e.

\[
\int \phi(t - k) \phi(t - j) dt = 0 \quad \text{for } j \neq k
\]

(25)

Using Eq. (25), Eq. (24) can be written as \( n \) simultaneous ODEs as

\[
\frac{1}{\Delta t^2} \sum_{k=j-N+2}^{j+N-2} \Omega_{j-k}^2 (I_0 u_k - I_1 \phi_k) - A_{11} \frac{d^2 u_j}{dx^2} + B_{11} \frac{d^2 \phi_j}{dx^2} - A_{13} \frac{d \psi_j}{dx} = 0, \quad j = 0, 1, \ldots, n - 1
\]

(26)

where \( N \) is the order of the Daubechies wavelet and \( \Omega_{j-k}^2 \) are the connection coefficients defined as

\[
\Omega_{j-k}^2 = \int \phi''(t - k) \phi(t - j) d\tau
\]

Similarly, for first order derivative \( \Omega_{j-k}^1 \) are defined as

\[
\Omega_{j-k}^1 = \int \phi'(t - k) \phi(t - j) d\tau
\]

For compactly supported wavelets, \( \Omega_{j-k}^2, \Omega_{j-k}^1 \) are non-zero only in the interval \( k = j - N + 2 \) to \( k = j + N - 2 \). The details for evaluation of connection coefficients for different orders of derivative is given in [31].

Similar to Eqs. (15) and (26), Eqs. (16)–(18) can be transformed as

\[
\frac{1}{\Delta t^2} \sum_{k=j-N+2}^{j+N-2} \Omega_{j-k}^2 (I_0 \psi_k + I_1 w_k) + A_{11} \frac{d u_j}{dx} - B_{11} \frac{d \phi_j}{dx} + A_{13} \psi_j - B_{55} \left( \frac{d^2 w_j}{dx^2} - \frac{d \phi_j}{dx} \right) - D_{55} \frac{d^2 \psi_j}{dx^2} = 0
\]

(29)

\[
\frac{1}{\Delta t^2} \sum_{k=j-N+2}^{j+N-2} \Omega_{j-k}^2 (I_0 w_k + I_1 \psi_k) - A_{55} \left( \frac{d w_j}{dx} - \frac{d \phi_j}{dx} \right) - B_{55} \frac{d^2 \psi_j}{dx^2} = 0
\]

(30)

\[
\frac{1}{\Delta t^2} \sum_{k=j-N+2}^{j+N-2} \Omega_{j-k}^2 (I_0 \phi_k - I_1 u_k) - A_{55} \left( \frac{d \phi_j}{dx} - \frac{d \psi_j}{dx} \right) - B_{55} \frac{d^2 \phi_j}{dx} + B_{11} \frac{d \psi_j}{dx} = 0
\]

(31)

The forced boundary condition associated with the governing differential equation given by Eq. (15) is

\[
A_{11} \frac{d \psi_j}{dx} - B_{11} \frac{d \phi_j}{dx} + A_{13} \psi_j = P_j, \quad j = 0, 1, \ldots, n - 1
\]

(32)

Using Eqs. (33) and (25), Eq. (32) can be transformed to

\[
A_{11} \frac{d u_j}{dx} - B_{11} \frac{d \phi_j}{dx} + A_{13} \psi_j = P_j, \quad j = 0, 1, \ldots, n - 1
\]

(34)

The force boundary conditions associated with the governing Eqs. (16)–(18) are

\[
A_{55} \frac{d \psi_j}{dx} + B_{55} \frac{d \psi_j}{dx} - A_{55} \phi_j = V_j
\]

(35)

\[
-A_{55} \frac{d u_j}{dx} + D_{55} \frac{d \psi_j}{dx} - B_{11} \psi_j = M_j
\]

(36)

\[
B_{55} \frac{d \psi_j}{dx} + D_{55} \frac{d \psi_j}{dx} - B_{55} \phi_j = Q_j, \quad j = 0, 1, \ldots, n - 1
\]

(37)

Similar to Eqs. (32) and (34), the above Eqs. ((35)–(37)) can be transformed as

\[
A_{55} \frac{d \psi_j}{dx} + B_{55} \frac{d \psi_j}{dx} - A_{55} \phi_j = V_j
\]

(38)

\[
-A_{55} \frac{d u_j}{dx} + D_{55} \frac{d \phi_j}{dx} - B_{11} \psi_j = M_j
\]

(39)

\[
B_{55} \frac{d \psi_j}{dx} + D_{55} \frac{d \psi_j}{dx} - B_{55} \phi_j = Q_j, \quad j = 0, 1, \ldots, n - 1
\]

(40)
While dealing with finite length data sequence, problems arise at the boundaries. It can be observed from the ODEs given by Eq. (26) that certain coefficients \(u_j\) near the vicinity of the boundaries \((j = 0 \text{ and } j = n - 1)\) lie outside the time window \([0, t_f]\) defined by \(j = 0, 1, \ldots, n - 1\). Several approaches like capacitance matrix methods \([3,32]\), penalty function methods for treating boundaries are reported in the literature. In this paper, first a circular convolution method is adopted assuming periodicity of the solution. Next, the wavelet based extrapolation scheme \([19–21]\) is implemented for solution of boundary value problems. This approach allows treatment of finite length data and uses polynomial to extrapolate coefficients at boundaries either from interior coefficients or boundary values. The method is particularly suitable for approximation in time for the ease to impose initial values. However, either of the above methods converts the ODEs given by Eqs. (26), (29)–(31) to a set of coupled ODEs. These ODEs are decoupled and solved using methods very similar to FSFEM. The details of the formulation is presented in the later sections. The periodic WSFE solution encounters all the problems of FSFEM in time domain analysis as discussed earlier. However, the periodic formulation allows the derivation of spectrum and dispersion relation and also the relation between the transformed ODEs in WSFEM with those in FSFEM. This leads to the direct use of WSFE for frequency domain analysis similar to FSFE.

4. Periodic boundary conditions

Eqs. (26), (29)–(31) of previous section, give \(n\) coupled ODEs each, which are to be solved for \(u_j (w_j, \phi_j, \psi_j)\) using method described later. For numerical implementation, we can deal with only finite sequence. In other words, \(u(x, t)\) and hence \(u_j\) are only known in the interval \([0, t_f]\) and \(j = 0 \text{ to } n - 1\). In Eqs. (26), (29)–(31) the ODEs corresponding to \(j = 0 \text{ to } N - 2\), contain coefficients \(u_j (w_j, \phi_j, \psi_j)\) that lie outside the \([0, t_f]\). Similarly, on the other boundary, for \(j = (n - 1) - N + 2 \text{ to } j = (n - 1)\) same problem exists.

One approach is to assume the function \(u(x, t)\) (similarly \(w(x, t), \phi(x, t)\) and \(\psi(x, t)\)) to be periodic in time, with time period \(t_p\). Thus the unknown coefficients on LHS are taken as

\[

t_{-1} = u_{n-1} \\
t_{-2} = u_{n-2} \\
\vdots \\
t_{N+2} = u_{n-N+2}
\]

Similarly the unknown coefficients on RHS, i.e. \(u_{n}, u_{n+1}, \ldots, u_{n+N-2}\) are equal to \(u_0, u_1, \ldots, u_{N-2}\), respectively. With the above assumption, the coupled ODEs given by Eq. (26) can be written in matrix form as

\[
\begin{align*}
[\mathbf{A}^2](I_0\{u_j\} - I_1\{\phi_j\}) - A_{11}\left\{\frac{d^2u_j}{dx^2}\right\} + B_{11}\left\{\frac{d^2\phi_j}{dx^2}\right\} \\
- A_{13}\left\{\frac{d\psi_j}{dx}\right\} = 0
\end{align*}
\]

where \(A^2\) are \(n \times n\) circulant connection coefficient matrices for second-order derivative and have the form

\[
A^2 = \frac{1}{\Delta t^2} \begin{bmatrix}
\Omega_0 & \Omega_1 & \cdots & \Omega_{N-2} & \cdots & \Omega_{N} \\
\Omega_1 & \Omega_0 & \cdots & \Omega_{N-3} & \cdots & \Omega_2 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
\Omega_{N-1} & \Omega_{N-2} & \cdots & \Omega_2 & \cdots & \Omega_0
\end{bmatrix}
\]

The connection coefficient matrix \(A^1\), for first-order derivative has a similar form. For a circulant matrices \(A^1\) (and \(A^2\)) \([33]\), the eigenvalues \(\lambda_j\) are

\[
\lambda_j = \frac{1}{\Delta t} \sum_{k=-N+2}^{N-2} \Omega_k^1 e^{2\pi i k j / n}, \quad j = 0, 1, \ldots, n - 1
\]

where \(i = \sqrt{-1}\) and the corresponding orthonormal eigenvector \(w_j, j = 0, 1, \ldots, n - 1\) are

\[
(w_j)_k = \frac{1}{\sqrt{n}} e^{-2\pi i j k / n}, \quad k = 0, 1, \ldots, n - 1
\]

For \(A^1\), \(\Omega_p^1 = -\Omega_{N-p}^1\) for \(p = 1, 2, \ldots, N-2\) and \(\Omega_0^1 = 0\) and we can write \(\lambda_j = i\beta_j\) where

\[
\beta_j = -\frac{2}{\Delta t} \sum_{k=1}^{N-2} \Omega_k^1 \sin\left(\frac{2\pi k j}{n}\right), \quad j = 0, 1, \ldots, n - 1
\]

It can be seen from the above derivations that the wavelet coefficients of first and second derivatives can be obtained as

\[
\{\dot{u}_j\} = A^1\{u_j\}
\]

\[
\{\ddot{u}_j\} = A^2\{u_j\}
\]

The second derivative can also be written as

\[
\{\ddot{u}_j\} = A^1\{\dot{u}_j\}
\]

Substituting Eq. (47) in Eq. (49) we get

\[
\{\ddot{u}_j\} = [A^1]^2\{u_j\}
\]

Thus though the second-order connection coefficient matrices \(A^2\) can be evaluated independently \([31]\), they can also written as

\[
A^2 = [A^1]^2
\]

The above modification is done as this form helps in imposing the initial conditions for non-periodic solution discussed later. Thus the Eq. (42) can be written as
\[ [A^1]^2 \{I_0 \{u_j\} - I_1 \{\phi_j\} \} - A_{11} \left\{ \frac{d^2 u_j}{dx^2} \right\} + B_{11} \left\{ \frac{d^2 \psi_j}{dx^2} \right\} - A_{13} \left\{ \frac{d \psi_j}{dx} \right\} = 0 \] (52)

As discussed earlier, the spectral element formulation, presented at the later part of the paper, involves eigenvalue analysis. This is done to diagonalize the matrix in Eq. (52) and decouple the ODEs. For periodic boundary condition, these eigenvalues are known analytically and hence decreases the computational cost. Thus, the matrix \( A^1 \) can be written as
\[
A^1 = \Phi \Pi \Phi^{-1} \tag{53}
\]
where \( \Pi \) is the diagonal matrix containing the diagonal terms \( i\beta_j \), and \( \Phi \) is the eigenvector matrix. Using Eq. (53), Eq. (52) can be decoupled and written as
\[
-I_0 \beta_j^2 \hat{u}_j + I_1 \beta_j^2 \hat{\phi}_j - A_{11} \frac{d^2 \hat{u}_j}{dx^2} + B_{11} \frac{d^2 \hat{\phi}_j}{dx^2} - A_{13} \frac{d \hat{\psi}_j}{dx} = 0 \tag{54}
\]
where \( \hat{u}_j \) and similarly other transformed displacements are
\[
\hat{u}_j = \Phi^{-1} u_j \tag{55}
\]
Following the above steps, the other three governing differential equations (16)–(18) can be transformed as
\[
-I_2 \beta_j^2 \hat{\phi}_j - I_4 \beta_j^2 \hat{\psi}_j + A_{13} \frac{d \hat{u}_j}{dx} - B_{13} \frac{d \hat{\phi}_j}{dx} + A_{43} \hat{\psi}_j
\]
\[
= B_{35} \left( \frac{d^2 \hat{\psi}_j}{dx^2} - \frac{d \hat{\phi}_j}{dx} \right) - D_{35} \frac{d \hat{\psi}_j}{dx} = 0 \tag{56}
\]
\[
-I_0 \beta_j^2 \hat{\psi}_j - I_3 \beta_j^2 \hat{\phi}_j - A_{35} \left( \frac{d \hat{u}_j}{dx} - \hat{\phi}_j \right) - B_{35} \frac{d \hat{\psi}_j}{dx} = 0 \tag{57}
\]
\[
-I_2 \beta_j^2 \hat{\phi}_j - I_4 \beta_j^2 \hat{\psi}_j - A_{55} \left( \frac{d \hat{\psi}_j}{dx} - \hat{\phi}_j \right) - B_{55} \frac{d \hat{\psi}_j}{dx} = 0 \tag{58}
\]
The form of the transformed equations (54) and (56)–(58) is same as those in FSFEM and thus the remaining part of WSFE formulation for composite beam will be exactly similar to FSFEM formulation described in [28].

5. Frequency domain analysis

Though periodic WSFE solution encounters all the problems of FSFEM in time domain analysis it allows the derivation of a relation between the transformed ODEs in WSFEM with those in FSFEM. This leads to the direct use of WSFE for frequency domain analysis similar to FSFEM.

For periodic solution, the wavelet transformation can be written as the matrix equation [1]
\[
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_{n-1}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & \cdots & \varphi_{N-2} & \varphi_{N-1} \\
\varphi_1 & 0 & 0 & \cdots & \varphi_2 & \varphi_3 \\
\varphi_2 & \varphi_1 & 0 & \cdots & \varphi_3 & \varphi_4 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\varphi_{N-2} & \varphi_{N-3} & \varphi_{N-4} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \varphi_{N-3} & \varphi_{N-2}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\hat{u}_2 \\
\vdots \\
\hat{u}_{n-1}
\end{bmatrix}
\]
(59)

where \( \varphi_j \) are the values of \( u(x, \tau) \) and \( \varphi(\tau) \) at \( \tau = j \). As described in [1], for such circulant matrix the Eq. (59) can be replaced by a convolution relation which can be written as
\[
\{ \tilde{U}_j \} = \{ \tilde{K}_{ij} \cdot \tilde{u}_j \} \tag{60}
\]
\[
\{ \tilde{u}_j \} = \{ \tilde{U}_j / \tilde{K}_{ij} \} \tag{61}
\]
where \( \{ \tilde{U}_j \}, \{ \tilde{u}_j \} \) are FFT of \( \{ U_j \}, \{ u_j \} \), respectively (similar relations hold for other displacements). \( \tilde{K}_{ij} \) is FFT of first column \( K_{ij} = \{ 0 \varphi_1 \varphi_2 \ldots \varphi_{N-2} \ldots 0 \} \) of the scaling function matrix in Eq. (59). Similarly the connection coefficient matrix \( [A^1] \) is also a circulant matrix and thus Eq. (42) can be written as
\[
I_0 \{ \tilde{K}_{ij} \cdot \tilde{u}_j \} - I_1 \{ \tilde{K}_{ij} \cdot \tilde{\phi}_j \} - A_{11} \left\{ \frac{d^2 \tilde{u}_j}{dx^2} \right\} + B_{11} \left\{ \frac{d^2 \tilde{\phi}_j}{dx^2} \right\} - A_{13} \left\{ \frac{d \tilde{\psi}_j}{dx} \right\} = 0 \tag{62}
\]
where \( \tilde{K}_{ij} \) are the FFT coefficients of the first column of \( [A^1] \) given as \( K_{ij} = \{ \Omega_1^i \Omega_1^{i-1} \ldots \Omega_{N-2}^{i} \ldots \Omega_1^{i} \} \). It can be easily shown that the FFT coefficients \( \tilde{K}_{ij} \) are equal to the eigenvalues \( i\beta_j \) of the matrix \( [A^1] \).

Substituting Eq. (61) in Eq. (62) and multiplying with \( \tilde{K}_{ij} \) on both side we get
\[
I_0 \{ \tilde{K}_{ij} \cdot \tilde{U}_j \} - I_1 \{ \tilde{K}_{ij} \cdot \tilde{\phi}_j \} - A_{11} \left\{ \frac{d^2 \tilde{U}_j}{dx^2} \right\} + B_{11} \left\{ \frac{d^2 \tilde{\phi}_j}{dx^2} \right\} - A_{13} \left\{ \frac{d \tilde{\psi}_j}{dx} \right\} = 0, \quad j = 0, 1, \ldots, n - 1 \tag{63}
\]
\[
-I_0 \beta_j^2 \tilde{U}_j + I_1 \beta_j^2 \tilde{\phi}_j - A_{11} \left\{ \frac{d^2 \tilde{U}_j}{dx^2} \right\} + B_{11} \left\{ \frac{d^2 \tilde{\phi}_j}{dx^2} \right\} - A_{13} \left\{ \frac{d \tilde{\psi}_j}{dx} \right\} = 0 \tag{64}
\]
In FSFE, the transformed ODEs are of same form except $\beta_j$ are replaced with $\omega_j$

$$- I_0 \omega_j^2 \ddot{U}_j + I_1 \omega_j^2 \ddot{\Phi}_j = A_{11} \left\{ \frac{d^2 U_j}{dx^2} \right\} + B_{11} \left\{ \frac{d^2 \Phi_j}{dx^2} \right\}$$

$$- A_{13} \left\{ \frac{d \Psi_j}{dx} \right\} = 0, j = 0, 1, \ldots, n - 1$$

(65)

where

$$\omega_j = \frac{2\pi f_j}{n \Delta t}$$

(66)

It can be shown that for a given sampling rate $\Delta t$, $\beta_j$ exactly matches $\omega_j$ up to a certain fraction of Nyquist frequency $f_{nyq} = \frac{1}{2\Delta t}$. This fraction, $p_N = f / f_{nyq}$ is dependent on the order of the basis $N$ and is more for higher order basis. Thus similar to FSFE, WSFE can be used directly for studying frequency dependent characteristics like spectrum and dispersion relation but up to a certain fraction of $f_{nyq}$. In Fig 2(a), $\omega_j$ and $\beta_j$ are plotted with respect to the fraction $p_N = f / f_{nyq}$ for different orders of basis. It can be seen that $\beta_j$ and $\omega_j$ matches exactly up to a fraction $p_N = 0.6$ for $N = 22$ which decreases with decrease in $N$ and is $\approx 0.36$ for $N = 6$. Wavenumbers and group speeds that are obtained directly in FSFE are functions of $\omega_j$. Similarly, the wavenumbers and group speeds extracted from WSFEM will be functions of $\beta_j$. Thus other parameters remaining same, these frequency dependent characteristics (wavenumber and group speeds) obtained from WSFEM and FSFEM will be exactly equal within the frequency range (given by $f_{Nyq} = \frac{f_{Nyq}}{N}$) where $\beta_j$ and $\omega_j$ are equal. Beyond this frequency range $f_N$, WSFEM will give spurious dispersions as there $\beta_j$ differ largely from $\omega_j$. In addition, for accurate simulation with WSFEM, the sampling rate $\Delta t$ should be adjusted such that the frequency content of the load is within the above mentioned allowed frequency range. Thus, this study also helps a-priori determination of the sampling rate required depending on the frequency content of excitation loads and order of bases. This has been explained with numerical experiments later.

6. Non-periodic boundary condition

For non-periodic solution the boundaries are treated using wavelet extrapolation method for Daubechies compactly supported wavelets. The detail of the formulation is given in [19–21]. In brief, this method uses a polynomial of order $p = N/2$ to extrapolate the values at the boundaries. Since, in this work the wavelets are used in time, the unknown coefficients on the LHS (i.e. $u_{-1}, u_{-2}, \ldots, u_{-N+2}$) are extrapolated from the initial values. The coefficients $u_0, u_{N+1}, \ldots, u_{N+2}$ on RHS are extrapolated from the known coefficients $u_{(N-1)-p+1}$, $u_{(N-1)-p+2}$, $\ldots, u_{N-1}$.

As discussed in the earlier work by authors [18], after treating the boundaries, the ODEs given by Eq. (26) can be written in matrix form, similar to Eq. (52)

$$[I^1]^2 \left\{ I_0 \{ u_j \} - I_1 \{ \phi_j \} \right\} - A_{11} \left\{ \frac{d^2 u_j}{dx^2} \right\} + B_{11} \left\{ \frac{d^2 \phi_j}{dx^2} \right\}$$

$$- A_{13} \left\{ \frac{d \psi_j}{dx} \right\} = 0$$

(67)

where $I^1$ is the first-order connection coefficient matrix. These coupled ODEs are similarly decoupled using eigenvalue analysis

$$I^1 = \Phi \tilde{\Pi} \Phi^{-1}$$

(68)

where $\tilde{\Pi}$ is the diagonal eigenvalue matrix and $\Phi$ is the eigenvectors matrix of $I^1$. Let the eigenvalues be $i_{ij}$, then the decoupled ODEs corresponding to Eqs. (54) and (56)–(58) can be written as

Fig. 2. Comparison of $\omega_j$, $\beta_j$ and $\gamma_j$ for different order ($N$) of basis. (a) Real part and (b) imaginary part of $\gamma_j$.
\begin{align}
- I_{10}\tilde{u}_j + I_1\tilde{\phi}_j - A_{11} \frac{d^2\tilde{u}_j}{dx^2} + B_{11} \frac{d^3\tilde{\phi}_j}{dx^3} - A_{13} \frac{d\tilde{\psi}_j}{dx} = 0
\quad \text{(69)}
\end{align}

\begin{align}
- I_{25}\tilde{\psi}_j - I_{17}\tilde{\psi}_j + A_{13} \frac{d\tilde{\psi}_j}{dx} - B_{13} \frac{d^2\tilde{\phi}_j}{dx^2} + A_{33}\tilde{\psi}_j
- B_{55} \left( \frac{d^2\tilde{w}_j}{dx^2} - \frac{d^2\tilde{\phi}_j}{dx^2} \right) - D_{55} \frac{d^2\tilde{\psi}_j}{dx^2} = 0
\quad \text{(70)}
\end{align}

\begin{align}
- I_{10}\tilde{\psi}_j - I_1\tilde{\phi}_j - A_{55} \left( \frac{d^2\tilde{w}_j}{dx^2} - \frac{d^2\tilde{\phi}_j}{dx^2} \right) - B_{55} \frac{d\tilde{\psi}_j}{dx}
+ B_{11} \frac{d^2\tilde{u}_j}{dx^2} - D_{11} \frac{d^2\tilde{\phi}_j}{dx^2} + B_{11} \frac{d\tilde{\psi}_j}{dx} = 0
\quad \text{(72)}
\end{align}

These decoupled equations are similar as that for periodic solution Eqs. (54) and (56)–(58) except that, \( \beta_j \) is replaced by \( \gamma_j \).

Unlike \( \beta_j \) which is real, \( \gamma_j \) is complex. However, from numerical experiments it is seen that the real part of \( \gamma_j \) matches \( \beta_j \), which are plotted and compared for different order of basis in Fig. 2(a). The additional imaginary part of \( \gamma_j \) are plotted for different basis in Fig. 2(b). Thus the imposition of non-periodicity, introduces an additional imaginary part in \( \gamma_j \) while the real part is exactly equal to \( \beta_j \) associated with periodic analysis.

7. Spectral finite element formulation

The degrees of freedom associated with the element formulation is shown in Fig. 1(b). The element has 4 degrees of freedom per node, which are \( \tilde{u}_j, \tilde{w}_j, \tilde{\phi}_j \), and \( \tilde{\psi}_j \). From the previous sections, we get a set of ODEs (Eqs. (54) and (56)–(58) for composite beam with axial, shear, transverse and contractional modes, in a transformed wavelet domain. These equations are required to be solved for \( \tilde{u}_j, \tilde{w}_j, \tilde{\phi}_j, \tilde{\psi}_j \) and the actual solutions \( u(x,t), w(x,t), \phi(x,t), \psi(x,t) \) are obtained using inverse wavelet transform. For finite length data, the wavelet transform and its inverse can be obtained using a transformation matrix [34]. For periodic solution the transformation matrix is given by Eq. (59). For non-periodic solution, required for finite length structures, the transformation matrix needs to be modified using wavelet extrapolation technique and this was described in previous section.

It can be seen that the transformed ODEs have a form which is similar to that in FSFEM [28]. Thus, WSFE can be formulated following the same method as for FSFE formulation and only \( \omega_j \) are replaced by \( \beta_j \). In this section, the subscript \( j \) is dropped hereafter for simplified notations and all the following equations are valid for \( j = 0, 1, \ldots, n - 1 \).

The exact interpolating functions for an element of length \( L \), obtained by solving Eqs. (54) and (56)–(58), respectively are

\[ \{ \tilde{u}(x), \tilde{w}(x), \tilde{\phi}(x), \tilde{\psi}(x) \}^T = [R] \{ \Theta \} \{ a \} \quad \text{(73)} \]

where \([\Theta]\) is a diagonal matrix with the diagonal terms \( e^{-ik_1x}, e^{-ik_2(x-L)}, e^{-ik_3(x-L)}, e^{-ik_4(x-L)} \) and \([R]\) is a \( 4 \times 8 \) amplitude ratio matrix for each set of \( k_1, k_2, k_3 \) and \( k_4 \).

\[ [R] = \begin{bmatrix}
R_{11} & \ldots & R_{18} \\
R_{21} & \ldots & R_{28} \\
R_{31} & \ldots & R_{38} \\
R_{41} & \ldots & R_{48}
\end{bmatrix} \quad \text{(74)} \]

\( k_1, k_2, k_3 \) and \( k_4 \) are obtained by substituting Eq. (73) in Eqs. (54) and (56)–(58). This gives the characteristic equation

\[
\begin{bmatrix}
(k^2 - k_0^2) & 0 & \left( \frac{\beta c_k}{k_0} - \frac{\beta c_k}{k_0} \right) & \frac{\beta c_k}{k_0} \\
0 & (k^2 - k_0^2) & -i \frac{\beta c_k}{k_0} & \frac{\beta c_k}{k_0} \\
\left( \frac{\beta c_k}{k_0} \right) & -i \frac{\beta c_k}{k_0} & (s_1^2 - 1 - \frac{\beta c_k}{k_0}^2) & -i \frac{\beta c_k}{k_0} \\
\frac{\beta c_k}{k_0} & i \frac{\beta c_k}{k_0} & \frac{\beta c_k}{k_0} & (s_2^2 - \frac{\beta c_k}{k_0}^2)
\end{bmatrix}
\times [R] \{ a \} = 0
\quad \text{(75)}
\]

where \( k_0 = \beta c_0, k_1 = \beta c_0, k_2 = \beta c_s \) and \( c_0 = \sqrt{A_{11}/I_0}, c_s = \sqrt{A_{55}/I_0} \). In Eq. (75), \( r \) and \( s_2 \) are non-dimensional axial-flexural coupling parameters due to the stiffness and the material asymmetry, respectively; \( s_1 \) is a non-dimensional flexural-shear coupling parameter. They are expressed as \( r = \sqrt{B_1^2/(A_1D_{11})} \), \( s_1 = \beta \sqrt{I_2/A_{55}} \) and \( s_2 = \sqrt{I_1/(I_0I_2)} \). \( k_1, k_2, k_3 \) and \( k_4 \) are obtained by solving the characteristic equation obtained by equating the determinant of the \( 4 \times 4 \) matrix in Eq. (75) to zero and the corresponding \([R]\) is obtained using singular value decomposition of the matrix. This method of determining wavenumbers and corresponding amplitude ratios was developed to formulate FSFE for graded beam with Poisson’s contraction in Ref. [35]. \( k_1, k_2, k_3 \) and \( k_4 \) corresponds to the four modes, i.e., axial, transverse, shear and contraction, respectively. As discussed earlier, \( \beta \) matches exactly with \( \omega \) up to a certain fraction of \( \omega_{nyq} \) and hence \( k_1, k_2, k_3, k_4 \) will match the wavenumbers \( (k_{f1}, k_{f2}, k_{f3}, k_{f4}) \) in FSFEM. Therefore these wavenumbers in WSFEM can be used for obtaining the spectrum and dispersion relations. This is explained in detail with numerical examples in the later section.
For shear and contractional modes, the wavenumbers become zero for certain \( \omega_p \), thus rendering the group speed equal to zero and the phase speed escapes to infinity. These frequencies are called cut-off frequencies [9]. In FSFEM, these frequencies are obtained by substituting \( k = 0 \) in the characteristic equation and solving for \( \omega_p \). In WSFEM, these cut-off frequencies are obtained similarly by solving for \( \gamma \) in place of \( \omega_p \). The expression of cut-off frequencies for shear and contractional modes are 
\[
\frac{4\pi x}{L} \quad \text{and} \quad \frac{4\pi x}{L(1-x)}
\]
respectively [28].

Here, \( \{a\} = \{A, B, C, D, E, F, G, H\} \) are the unknown constants \( \{a\} \) are known from the above equations, they can be used effectively to study frequency dependent wave parameters. As discussed in earlier sections, WSFEM can be used to obtain the frequency dependent wave characteristics similar to FSFEM, but up to a fraction of Nyquist frequency \( f_{nyq} \). Here, first WSFEM is used to study the spectrum and dispersion relations for all the four modes. Next, non-periodic WSFEM is used to simulate wave propagation in finite length cantilever beam due to broadband impulse load. The results are validated with 1-D FE results. Comparisons are also provided with WSFEM results to show the advantages of WSFEM for time domain analysis of wave propagation in finite length structures. Finally, periodic WSFEM is used to study the response due to sinusoidal modulated pulse applied to an infinite beam. This example helps to capture graphically all the non-dispersive propagating modes.

All the numerical experiments presented are performed on a AS4/3501-6 graphite-epoxy composite beam with four plies. The beam has a depth of \( h = 0.01 \) m and width \( b = 0.01 \) m and the material properties are given in Table 1. The order of Daubechies scaling function basis used is \( N = 22 \) unless otherwise mentioned.

### 8.1. Spectrum and dispersion relation

In Fig. 3, the wavenumbers for a [0\text{\_}] AS4/3501-6 graphite-epoxy beam, computed using periodic WSFEM \( (k_w) \) with \( N = 22 \) are plotted for all the four modes i.e axial, flexural, shear and contractional. It can be seen that the shear and contractional modes have a cut-off frequency and will propagate for loading with frequencies higher than these cut-off frequencies. Before the cut-off frequency, the wavenumbers corresponding to the shear and contractional modes are imaginary and become real at frequencies higher than the cut-off.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Properties of AS4/3501-6 graphite-epoxy beams</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material properties</td>
<td></td>
</tr>
<tr>
<td>( E_{11}, ) GPa</td>
<td>141.9</td>
</tr>
<tr>
<td>( E_{22}, ) GPa</td>
<td>9.78</td>
</tr>
<tr>
<td>( G_{12} = G_{13}, ) GPa</td>
<td>6.13</td>
</tr>
<tr>
<td>( G_{23}, ) GPa</td>
<td>4.80</td>
</tr>
<tr>
<td>( \nu_{12} )</td>
<td>0.42</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1449 kg/m(^3)</td>
</tr>
</tbody>
</table>
frequencies analytically. For \([0,4]\) ply orientation, these cut-off frequencies for shear and contractional modes are approximately 100 kHz and 162 kHz, respectively. In composites the contractional cut-off frequency is always greater than shear cut-off frequency as \(A_{33} > A_{55}\). For all the modes, comparisons are provided with those obtained using FSFEM (\(k_f\)). The sampling rate is \(n_t = 1\) and corresponding Nyquist frequency \(f_{nyq} = 500\) kHz. As discussed earlier, it can be seen that all the \(k_w\) matches exactly with \(k_f\) up to a certain fraction \(p_N\) (approximately 0.6 here) of \(f_{nyq}\) beyond which, we can see spurious dispersion. Thus similar to FSFEM, WSFEM can be accurately used for frequency domain analysis but up to the fraction \(p_N\) of \(f_{nyq}\). This \(p_N\) giving the allowable frequency range \(f_N = p_N f_{nyq}\) depends only on the order of basis \(N\). In Fig. 4, the percentage error in form of \(|k_f - k_w|/|k_f|\) is plotted with respect to the fraction \(f/f_{nyq}\) for different order of basis functions. It can be seen that the calculated error is negligible and almost equal to zero up to a certain \(f/f_{nyq}\) and then increases quite steeply. For a given \(N\), the fraction \(p_N\) as defined earlier is the highest frequency fraction up to which the error remains negligible and this can be derived numerically from Fig. 4 for different \(N\). A consequence of this study is the a-priori determination of sampling rate \(\Delta t\) required for accurate analysis with WSFEM. In WSFEM, the \(\Delta t\) should be such that the frequency content of the load should be within the allowable frequency range \(f_N\) to avoid the introduction of spurious dispersion in the analysis.

Next, to study how the wave packets travel at different frequencies, the dispersion relations are plotted for axial, flexural, shear and contractional modes. In Fig. 5, the dispersion relation derived from WSFEM with \(N = 22\) is plotted for \(\Delta t = 1\) \(\mu s\) or \(f_{nyq} = 500\) kHz and compared to that obtained from FSFEM. In this figure, the non-dimensional group speeds \(C_g/C_0\) where \(C_g = \text{real}(d\xi/dk)\) and \(C_0 = \sqrt{EA/\rho A}\) are plotted with respect to frequency. One significant difference between higher order and elementary beam theory is the presence of the shear and contractional propagating modes. These two modes start to propagate only after the respective cut-off frequencies. Similar to spectrum relation, WSFEM predicts the exact speeds up to almost \(f_N\) and is approximately equal to 150 kHz here for \(N = 22\).

### 8.2. Cantilever beam under tip impulse load

Here, non-periodic WSFEM is used for analysis of wave propagation in a finite length cantilever beam due to unit broad-band impulse load applied at the free
end. As mentioned earlier for analysis of such finite length structure, non-periodic WSFEM is adopted as it allows imposition of boundary condition and remove wrap around problem. The impulse load shown in Fig. 6 is applied on to the structure and this load has a duration of 50 μs and frequency content of 44 kHz. In Fig. 7(a), the tip axial velocities in the undamped beam of length \( L = 0.5 \text{ m} \) and ply orientation \([0/90_2]\) due the impulse load applied at the tip in axial direction are plotted. The results obtained using WSFEM is compared with 1-D FE results and they match very well. In Fig. 7(b), the tip transverse velocities in the undamped beam of length \( L = 0.25 \text{ m} \) and ply orientation \([0/90_2]\) due the impulse load applied at the tip in transverse direction are plotted. Even here, the WSFEM and 1-D FE results compare well. Both first-order shear deformation (FSDT) and Euler–Bernoulli theory (EBT) are considered. The EBT solution are derived from the formulated FSDT solution by considering \( A_{55} = \infty \). WSFEM solutions are obtained with \( N = 22 \) and sampling rate \( \Delta t = 2 \mu s \). The time window considered is \( T_w = 512 \mu s \) and as said earlier, unlike FSFEM, the accuracy of WSFEM solution is independent of \( T_w \). In addition, WSFEM requires a single element to predict the above result which validates the exactness of the formulated dynamic stiffness matrix. FE solution are obtained using 2000 1-D beam element and Newmarks time integration scheme with time step of 1 μs. In addition, it can be seen from Fig. 7(b) that effect of neglecting transverse shear causes substantial reduction of transverse wave velocity.

In Figs. 8(a) and (b), the tip axial and transverse velocities in a undamped cantilever beam due to the tip unit impulse load applied in axial and transverse directions, respectively are plotted for different ply orientations. The velocities are simulated using WSFEM with \( N = 22 \), sampling rate \( \Delta t = 2 \mu s \) and \( T_w = 512 \mu s \). The length of the beam considered is \( L = 0.5 \text{ m} \) for measuring the axial velocity and is \( L = 0.25 \text{ m} \) for transverse velocity. The three different ply orientations considered are \([0_4]_4\), \([0_2/60_1]_2\) and \([0_2/90_3]\). It can be seen from the figures that both the axial and transverse velocities are highest for \([0_4]\) ply layup while least for \([0_2/90_3]\) ply layup.

Next, the WSFEM and FSFEM solutions are compared to emphasize the advantages of WSFEM for wave propagation analysis in finite length structures. In Fig. 9, the tip axial velocities in a \([0_2/60_1]\) cantilever beam with \( L = 0.25 \text{ m} \) due to the impulse load applied at tip in transverse direction are plotted. Though WSFEM can be used effectively for analysis of undamped finite length structures, for comparison with FSFEM, which does not work for such cases, a damping of \( \eta = 0.1 \) (see Ref. [28]) is considered. The sampling rate \( \Delta t \) for both FSFEM and WSFEM is 2 μs. However, WSFEM results are obtained with \( T_w = 512 \mu s \) while FSFEM requires much higher \( T_w \) to remove the distortion due to
wrap around problem. It can be seen from Fig. 9 that FSFEM solution with $T_w = 16384 \mu s$ shows large distortions which decrease by increasing $T_w$ further to 32,768 $\mu s$.

8.3. Response to modulated sinusoidal pulse

To study the presence of different propagating coupled mode in an asymmetric beam, the responses to narrow banded sinusoidal pulse modulated at a high frequency are simulated using periodic WSFEM. For such loading the waves propagate non-dispersively. The load is applied at a point $C$ on an infinite beam (see Fig. 1(c)) and the velocities are measured at point $D$ at a distance $L$ from $C$. Here, a ply orientation of $[0/2/602]$ and $L = 2.0$ m is considered. The load is modulated at 200 kHz, such that the loading frequency is above the highest cut-off frequency which is

![Fig. 8](image.png)

Fig. 8. (a) Axial and (b) transverse tip velocities in a graphite-epoxy beam with different ply orientations due to tip impulse load applied in axial and transverse directions, respectively.

![Fig. 9](image.png)

Fig. 9. Axial tip velocities in a graphite-epoxy $[0/2/602]$ beam due to tip impulse load applied in transverse direction.

![Fig. 10](image.png)

Fig. 10. (a) Axial and (b) transverse velocities in a graphite-epoxy infinite $[0/2/602]$ beam due to modulated sinusoidal pulse at 200 kHz applied in transverse direction.
approximately equal to 165 kHz for contractional mode for this ply layup (see Ref. [28]). In Figs. 10(a) and (b), the axial and transverse velocities at C due to the pulse applied at D in transverse direction are plotted. It shows the presence of all the four propagating modes namely, axial, transverse, shear and contraction. Similarly, Figs. 11(a) and (b), show the axial and transverse velocities at C due to the pulse applied at D in axial direction. In all the above experiments, simulations are done with \( N = 22 \) and \( \Delta t = 1 \mu s \). As discussed earlier, in WSFEM, the sampling rate \( \Delta t \) should be more than a certain mandated value depending on the excitation frequency and order of basis \( N \) for accurate simulation. This becomes more important for modulated sinusoidal loading since here the loading frequency is very high. As an example it can be said that for the above numerical experiments with loading frequency 200 kHz, WSFEM with \( \Delta t = 2 \mu s \) will fail to predict the actual response and spurious modes will appear in the solution. This is because for \( \Delta t = 2 \mu s \) \( (f_{nyq} = 250 \text{ kHz}) \) and \( N = 22 \), the allowable frequency range is \( f_N = p_N f_{nyq} = 150 \text{ kHz} \) which is lower than the excitation frequency of 200 kHz. However, FSFEM with \( \Delta t = 2 \mu s \) will effectively simulate the correct response. This is shown in Fig. 12, where the transverse velocity measured at C in \([0_2/60_2]\) infinite beam due to the 200 kHz modulated pulse applied at D in transverse direction (similar to Fig. 10(b)) are plotted. Here, the simulations are done with \( \Delta t = 2 \mu s \). As explained above, FSFEM is capable of simulating the accurate response (Fig. 12(a)) with \( \Delta t = 2 \mu s \) while WSFEM (Fig. 12(b)) shows spurious dispersions.

9. Conclusions

This paper presents the formulation and validation of a wavelet based spectral element for axial-shear-
contraction-flexural coupled wave propagation in a composite beam with different ply orientations. The spectral element method is shown to be an efficient alternative of FE analysis of wave propagation problems and decreases the computational cost substantially. The novelty of the spectral element developed here is that it uses wavelet transform to reduce the PDEs to ODEs unlike FFT in conventional spectral element formulation. Due to the localized nature of Daubechies wavelet basis functions the present method proves to be more efficient as it removes the wrap around problem associated with FSFEM for time domain analysis. In this paper, WSFEM is also used to study the frequency dependent spectrum and dispersion relations. Based on this frequency domain analysis, the sampling rate required in WSFEM depending on the excitation frequency and order of wavelet basis can be a-priori determined. Numerical experiments are performed to study the wave propagation due to broad-band impulse load and modulated sinusoidal pulse. The effect of different ply orientations and transverse shear on the wave velocities are also studied. Comparisons with FSFEM results are also presented in many cases to highlight the advantages and limitations of WSFEM.

References