Single Degree of Freedom (SDOF) system

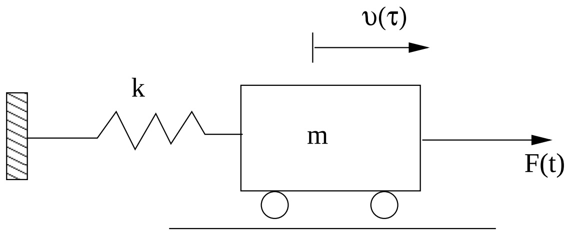

Figure 1: Undamped SDOF system

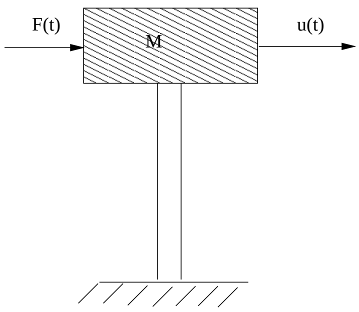

Figure 2: Example of overhead water tank that can be modeled as SDOF system

1. **Equation of motion (EOM)**
   Mathematical expression defining the dynamic displacements of a structural system. Solution of the expression gives a complete description of the response of the structure as a function of time

Derivation of EOM

1. Dynamic Equilibrium (Using D'Alembert's principle)
2. Principle of Virtual Work
3. Hamilton's principle (Using Lagrange's equation)

Dynamic Equilibrium

D'Alembert's principle states that a mass develops an inertial force proportional to its acceleration and opposing its motion. (See Figure 3)

$$m\ddot{u} + ku = F(t) \quad \text{Equation of Motion} \tag{1}$$

for $F(t) = 0$, the response is termed as free vibration and occurs due to initial excitation.

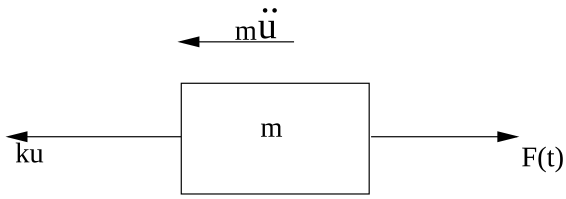

Figure 3: Dynamic force equilibrium

<u>Free Vibration</u>

$$
\begin{aligned}
& m\ddot{u} + ku = 0 \;\; \text{linear,homogeneous second order differential equation} \\
& \Rightarrow \ddot{u} + \frac{k}{m}u = 0 \\
& \Rightarrow \ddot{u} + \omega_n^2 u = 0 \quad \omega_n^2 = \frac{k}{m},\; \omega_n = \sqrt{\frac{k}{m}} \;\; \omega_n = \text{natural frequency}
\end{aligned} \tag{2}
$$

Solution of Equation 2 will be,

$$
\begin{aligned}
u(t) &= C_1 e^{\imath\omega_n t} + C_2 e^{-\imath\omega_n t} \\
&= C_1(\cos\omega_n t + \imath\sin\omega_n t) + C_2(\cos\omega_n t - \imath\sin\omega_n t) \\
&= (C_1 + C_2)\cos\omega_n t + \imath(C_1 - C_2)\sin\omega_n t
\end{aligned} \tag{3}
$$

Applying the initial conditions,

$$
\begin{aligned}
u(t)|_{@t=0} &= u_0 = C_1 + C_2 \\
\dot{u}(t)|_{@t=0} &= \dot{u}_0 = \imath\omega_n(C_1 - C_2)
\end{aligned} \tag{4}
$$

Substituting Equation 4 into Equation 3, we get,

$$
u(t) = u_0\cos\omega_n t + \frac{\dot{u}_0}{\omega_n}\sin\omega_n t \tag{5}
$$

Again, substituting,

$$
\begin{aligned}
u_0 &= A\cos\phi \\
\frac{\dot{u}_0}{\omega_n} &= A\sin\phi
\end{aligned} \tag{6}
$$

into Equation 5, we get,

$$
\begin{aligned}
u(t) &= A\cos\phi\cos\omega_n t + A\sin\phi\sin\omega_n t \\
&= A\cos(\omega_n t - \phi)
\end{aligned} \tag{7}
$$

where, A is the *amplitude* and ϕ is the *phase angle*

$$A = \sqrt{u_0^2 + \left(\frac{\dot{u}_0}{\omega_n}\right)^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{\dot{u}_0/\omega_n}{u_0}\right) \tag{8}$$

Free vibration of damped SDOF system

Modeling of damping is perhaps one of the most difficult task in structural dynamics. It is still a topic of research in advanced structural dynamics and is derived mostly experimentally.

Viscous Damping

The most common form of damping is viscous damping.

Equation of Motion

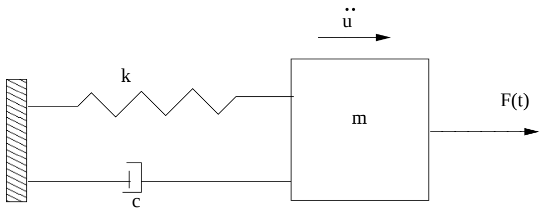

Figure 4: SDOF with viscous damping

$$\begin{aligned} & m\ddot{u} + c\dot{u} + ku = 0 \\ & \Rightarrow \ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0 \\ & \Rightarrow \ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = 0 \end{aligned} \tag{9}$$

where, $\xi = \frac{c}{2m\omega_n}$ is the *viscous damping factor.* Assuming a solution $u(t) = Ce^{st}$ and substituting in Equation 9, we get,

$$\begin{aligned} & s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \\ & \Rightarrow s = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} \\ & = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n \end{aligned} \tag{10}$$

Depending on the value of ξ, the nature of s and correspondingly $u(t)$ will be determined,

$$\begin{aligned} u(t) &= C_1 e^{(-\xi+\sqrt{\xi^2-1})\omega_n t} + C_2 e^{(-\xi-\sqrt{\xi^2-1})\omega_n t} \\ &= \left[C_1 e^{\sqrt{(\xi^2-1)}\omega_n t} + C_2 e^{-\sqrt{(\xi^2-1)}\omega_n t}\right] e^{-\xi\omega_n t} \end{aligned} \tag{11}$$

Case I *Under-damped system*, $0 < \xi < 1$
For $\xi < 1$, $s_1,\ s_2$ are complex numbers and given as,

$$s_1\ s_2 = \left(-\xi \pm \imath\sqrt{|\xi^2-1|}\right)\omega_n \tag{12}$$

Therefore,

$$u(t) = \left(C_1 e^{\imath\sqrt{|\xi^2-1|}\omega_n t} + C_2 e^{-\imath\sqrt{|\xi^2-1|}\omega_n t}\right)e^{-\xi\omega_n t} \tag{13}$$

Considering $\sqrt{|\xi^2-1|}\omega_n = \omega_d$, Equation 13 can be written as,

$$\begin{aligned} u(t) &= \left(C_1 e^{\imath\omega_d t} + C_2 e^{-\imath\omega_d t}\right)e^{-\xi\omega_n t} \\ &[(C_1+C_2)\cos\omega_d t + \imath(C_1-C_2)\sin\omega_d t]\,e^{-\xi\omega_n t} \end{aligned} \tag{14}$$

where, ω_d is referred as *damped natural frequency.* Substituting $(C_1 + C_2) = A\cos\phi$ and $\imath(C_1 - C_2) = A\sin\phi$ into Equation 14, we get,

$$u(t) = A\cos(\omega_d t - \phi)e^{-\xi\omega_n t} \tag{15}$$

Applying initial conditions as, $u(t)|_{@t=0} = u_0$ and $\dot{u}(t)|_{@t=0} = \dot{u}_0$, we get,

$$C_1 + C_2 = u_0 \quad \text{and} \quad \imath(C_1 - C_2) = \left[\frac{\dot{u}_0}{\omega_d} + \frac{u_0\xi}{\sqrt{1-\xi^2}}\right]$$

Thus for these initial conditions, the response can be written as,

$$u(t) = \left(u_0\cos\omega_d t + \left[\frac{\dot{u}_0}{\omega_d} + \frac{u_0\xi}{\sqrt{1-\xi^2}}\right]\sin\omega_d t\right)e^{-\xi\omega_n t} \tag{16}$$

Case II *Critically-damped system*, $\xi = 1$
Critical damping is the minimum damping required to stop the oscillations.

$$s_1,\ s_2 = -\omega_n$$

The solution is of the form,

$$u(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{17}$$

Even here, C_1 and C_2 can be obtained from the initial conditions given.
Case III *Over-damped system*, $\xi > 1$
There is no oscillatory motion in an over-damped system.

$$u(t) = (C_1 e^{\omega_d t} + C_2 e^{-\omega_d t})e^{-\xi\omega_n t} \tag{18}$$

For a over-damped system, higher the values of ξ, the slower the rate of the decay (See Figure 5).

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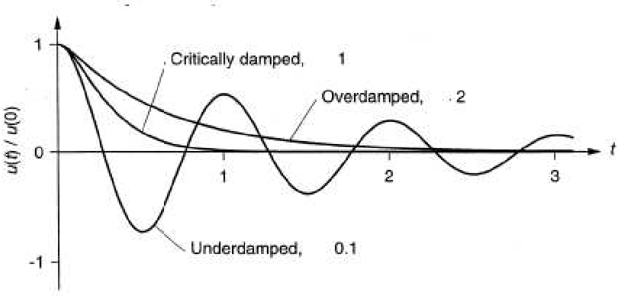

Figure 5: Free vibration of under-damped, critically damped and over-damped system