Response to Arbitrary Loading
In the previous lectures, forced response of under damped SDOF system were obtained for harmonic loading and periodic (non-harmonic) loading. Next, an analysis technique to obtain the response of under-damped SDOF due to arbitrary loading condition is derived here. Prior to deriving the technique, we need to know about impulse and response due to impulse.

**Impulse Response**
In a heuristic sense, *impulsive load* is one that acts for a very short duration. The above figure shows an *unit impulse* acting at $t = a$. The width $\epsilon \to 0$. Mathematically an unit impulse is represented by Dirac-delta $\delta(t - a)$ function defined as,

$$\delta(t - a) = 0 \forall \ t \neq a$$  \hspace{1cm} (1)\n
$$\int_{-\infty}^{\infty} \delta(t-a) dt = 1$$  \hspace{1cm} (2)\n
The integral in Equation 2 is non-dimensional and hence $\delta(t-a)$ has an unit s$^{-1}$. An impulse load having arbitrary magnitude of $F_0$ can be written as,

$$F(t) = F_0\delta(t-a)$$  \hspace{1cm} (3)\n
where, $F_0$ has the unit of force-sec like N-s, etc. The equation of motion of an under-damped single degree of freedom under impulse loading at $t = 0$ is given as,

$$m\ddot{u} + c\dot{u} + ku = F_0\delta(t)$$  \hspace{1cm} (4)\n
An impulse load acting at $t = 0$ results in an initial velocity and the response due to such loading can be thought as free vibration under the initial velocity. Applying Newton’s law, this initial velocity can be obtained in the following way,

$$F = \frac{d(mv)}{dt}$$
\[ F \, dt = mdv \]
\[
\int_0^\epsilon F \, dt = \int_0^\epsilon mdv
\]
\[
\int_0^\epsilon F_0 \delta(t-a)\, dt = \int_0^\epsilon mdv
\]
\[
F_0 = m[v(\epsilon) - v(0)] \quad \text{where} \quad \epsilon \to 0
\]
\[
v(\epsilon) = v(0+) = v(0) = \frac{F_0}{m}
\]

Substituting these initial conditions \((u(0) = 0 \text{ and } \dot{u}(0) = F_0/m)\) into the free vibration expression we get the response as,

\[
u(t) = \frac{F_0}{m\omega_d}e^{-\xi\omega_n t} \sin \omega_d t
\]

Thus, the response due to unit impulse is given and defined as,

\[
h(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n t} \sin \omega_d t
\]

Similarly, the response to unit impulse load applied at \(t = \tau\) can be written as,

\[
h(t - \tau) = \frac{1}{m\omega_d}e^{-\xi\omega_n(t-\tau)} \sin \omega_d(t - \tau) \quad t > \tau
\]

**Convolution/Duhamel Integral**

Let us consider an arbitrary loading as shown in Figure 2. The load can be thought as superposition of several impulse load. Let us consider one such impulse load which is part of the arbitrary load as shown in the figure. The impulse can be written as,

\[
F(t) = F(\tau) \Delta \tau \delta(t - \tau)
\]
The response due to this loading can be written as,

\[ u_\tau(t) = F(\tau) \Delta \tau h(t - \tau) \]  

(10)

Hence, the total response can be written as,

\[ u(t) = \sum F(\tau) \Delta \tau h(t - \tau) \]  

(11)

As \( \Delta \to 0 \), the summation can be replaced by integral and the response can be written as,

\[ u(t) = \int_0^t F(\tau) h(t - \tau) d\tau \]  

(12)

The integral in Equation 12 is called the convolution or Duhamel integral.