Extraction of wave characteristics from wavelet-based spectral finite element formulation

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Abstract

In this paper, a spectrally formulated wavelet finite element is developed and is used not only to study wave propagation in 1-D waveguides but also to extract the wave characteristics, namely the spectrum and dispersion relation for these waveguides. The use of compactly supported Daubechies wavelet basis circumvents several drawbacks of conventional FFT-based Spectral Finite Element Method (FSFEM) due to the required assumption of periodicity, particularly for time domain analysis. In this work, a study is done to use the formulated Wavelet-based Spectral Finite Element (WSFE) directly for such frequency domain analysis. This study shows that in WSFE formulation, a constraint on the time sampling rate is paced to avoid spurious dispersion being introduced in the analysis. Numerical experiments are performed to study frequency-dependent wave characteristics (dispersion and spectrum relations) in elementary rod, Euler–Bernoulli and Timoshenko beams. The effect of sampling rate on the accuracy of WSFE solution for both impulse and modulated sinusoidal loading with different frequency content is shown through different examples. In all above cases, comparison with FSFEM are provided to highlight the advantages and limitations of WSFE.

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1. Introduction

Wavelets have several properties which are encouraging their use for numerical solutions of partial differential equations (PDEs) [1–5]. Ref. [6] provides a review of wavelet techniques for solution of PDEs. The orthogonal, compactly supported wavelet basis of Daubechies [7,8] exactly approximates polynomial of increasingly higher order. These wavelet bases can provide accurate and stable representation of differential operations even in region of strong gradients or oscillations. In addition, the orthogonal wavelet basis have the inherent advantage of multi-resolution analysis over the traditional methods.

Numerical solution of elastic wave equations requires high accuracy in numerical differentiation and at the same time have larger spatial grids and time steps to make it computationally efficient. Wave propagation problems deal with loading that have very high-frequency content. Finite element (FE) formulation for wave propagation problems requires large system size to capture all the higher modes. Hence the element size has to be comparable to wavelengths, which are very small at high frequencies. These problems are usually solved in frequency domain using Fourier methods, which can in principle achieve high accuracy in numerical differentiation. One such method is Spectral Finite Element Method (SFEM) [9].

In conventional SFEM, first the governing PDE is transformed to frequency domain using FFT in time. For one-dimensional (1-D) structures, the governing PDE is reduced to a set of ODEs with constant coefficients, with frequency as a parameter. The resulting ODEs are much easier to solve than the original PDE. The ODEs are then usually solved exactly, which are used as interpolating functions for spectral element formulation. This results in exact mass distribution and dynamic stiffness matrix. Hence, in the absence of any discontinuity, one single element is sufficient to handle an 1-D waveguide of any length. This substantially reduces the system size and they are many orders smaller compared to conventional FE. First, the exact dynamic stiffness is used to determine the system transfer function (frequency-response function). This is then convolved with load. Next, inverse fast Fourier transform (IFFT) is used to get the time history of the response. Such FFT-based Spectral Finite Element (SFSE) for elementary rod [10], elementary beam [11,12], multiply connected (1-D) higher-order isotropic wave guides [13,14] are reported in literature. In the area of composite, FSF has been developed for Euler–Bernoulli beam [15] and Timoshenko beam [16]. FSFE formulation for wave propagation analysis in functionally graded beam is presented in [17].

The main drawback of Fourier-based spectral approach is that it cannot handle waveguides of short lengths. This is because, short length forces multiple reflections at smaller time scales. Since Fourier transforms are associated with a finite time window (that depends on time sampling rate), shorter lengths of waveguide do not allow the response to die down within the chosen time window, irrespective of the type of damping used in modelling. This forces the response to wrap around, that is the remaining part of the response beyond the chosen time window will start appearing first. This totally distorts the response. It is in such cases, compactly supported wavelets which have localised basis functions can be efficiently used for waveguides of short lengths. Different wavelet-based modelling techniques for simulation of wave propagation have been presented in [18–20].

WSFE formulation [21] is very similar to FSFEM, except that Daubechies scaling functions are used for approximation in time. This reduces the PDEs to coupled ODEs which are decoupled.
using eigenvalue analysis. The decoupled ODEs are then solved similarly as in FSFEM. The wavelet analysis can be adapted to a finite domain and initial values can be imposed using the wavelet extrapolation technique [22–24]. This removes the problem associated with “wrap around” due to the assumed periodicity of solutions in FSFEM and thus result in smaller time window for the same problem. For similar reason, WSFE can be used for analysis of undamped structures where FSFEM does not work. However, WSFE can also be formulated by considering periodic boundary condition and for this case the results are expected to be similar to those obtained using FSFEM.

Though FSFEM encounters several problems in time domain analysis of wave propagation, it is the only method used to study the various frequency-dependent characteristics of waves namely the spectrum and dispersion relation. In FSFEM the elastic wave equations are transformed to frequency domain using FFT. Thus, the spectrum (frequency dependence of wavenumber) and dispersion (frequency dependence of wave speeds) relations can be obtained directly from analysis in the transformed domain. However, this is not the case with wavelet transformed PDEs. Here, the transformation of variables gives a matrix equation, wherein the interpretation of the wave characteristics are not straight forward. The size of the equation depends on the number of sampling points (and hence the sampling rate). Analysis of the shorter waveguides always comes with a price. This comes in the form of constraint on the time sampling rate of the input signal. In this paper, a correspondence is established between the transformed ODEs in periodic WSFEM with those obtained in FSFEM. This study helps in using the formulated WSFE directly for such frequency domain analysis of wave propagation. The effect of imposing boundary conditions (through wavelet extrapolation technique) in the frequency domain for non-periodic solutions is also studied numerically.

It should be mentioned here that though in the present paper, WSFEM is developed for 1-D rods and beams, scope for further studies lies in developing such methods for wave propagation analysis of 2-D structures e.g. plates and shells. Such extension to higher-dimensional cases may be done similarly as FSFE formulation for 2-D semi-infinite plates [25]. Other approaches can also be studied for such 2-D cases, for example, the transformed ODEs can be solved using conventional FE techniques. This will help to handle arbitrary geometries and boundary conditions. Further, the method can be modified for solution of PDEs with variable coefficients (e.g. beam with varying cross-section), non-linear deformation and inelastic materials. However, the viability of such modifications can be assured only after detailed analysis.

In addition, in the present work, wave propagations due to simulated broad-band impulse load and narrow band-modulated sinusoidal (tone burst) load are studied. Such simulated loading has much lesser complexities than experimental loading as the later is always associated with noise. Thus, in these cases, denoising is a prerequisite for efficient wave propagation analysis. A continuous wavelet-based method for denoising in a falling weight impact test has been presented in [26]. In [27], a wavelet-based technique has been presented for inverse problems of measuring temperature from sideways heat equation using experimentally measured noisy data. Here, Meyer and Daubechies wavelets were used to approximate the time derivative. Such issues of denoising while working with experimental data will be studied in our future research.

The paper is organised as follows. In Section 2, a brief overview of the orthonormal bases of compactly supported wavelets are presented. In Sections 3–7, the details of wavelet-based spectral element formulation is given for isotropic Timoshenko beam with axial, transverse and shear
degrees of freedom (dof) at each node. In Section 8 various numerical experiments are presented. First, WSFEM is used to obtain the frequency-dependent parameters i.e. wavenumber and wave speeds for the elementary rod, Euler–Bernoulli and Timoshenko beam. Numerical examples are presented to show the effect of sampling rate on the accuracy of WSFE solution for both impulse and modulated sinusoidal loading with different frequency content. These studies will bring out the presence of spurious dispersion on using signals sampled using higher sampling rate. The paper ends with some important conclusions.

2. Daubechies compactly supported wavelets

In this section, a concise review of orthogonal basis of Daubechies wavelets [7,8] is provided. Wavelets, $\psi_{j,k}(t)$ forms compactly supported orthonormal basis for $L^2(\mathbb{R})$. The wavelets and associated scaling functions $\varphi_{j,k}(t)$ are obtained by translation and dilation of single functions $\psi(t)$ and $\varphi(t)$, respectively.

$$\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k), \quad j, k \in \mathbb{Z}, \quad \text{(1)}$$

$$\varphi_{j,k}(t) = 2^{j/2} \varphi(2^j t - k), \quad j, k \in \mathbb{Z}. \quad \text{(2)}$$

The scaling functions $\varphi(t)$ are derived from the dilation or scaling equation,

$$\varphi(t) = \sum_k a_k \varphi(2t - k) \quad \text{(3)}$$

and the wavelet function $\psi(t)$ is obtained as

$$\psi(t) = \sum_k (-1)^k a_{1-k} \varphi(2t - k), \quad \text{(4)}$$

$a_k$ are the filter coefficients and they are fixed for specific wavelet or scaling function basis. For compactly supported wavelets only a finite number of $a_k$ are non-zero. The filter coefficients $a_k$ are derived by imposing certain constraints on the scaling functions which are as follows. (1) The area under scaling function is normalised to one.

$$\int_{-\infty}^{\infty} \varphi(t) \, dt = 1. \quad \text{(5)}$$

(2) The scaling function $\varphi(t)$ and its translates are orthonormal

$$\int_{-\infty}^{\infty} \varphi(t) \varphi(t + k) \, dt = \delta_{0,k}, \quad k \in \mathbb{Z} \quad \text{(6)}$$

and (3) wavelet function $\psi(t)$ has $M$ vanishing moments

$$\int_{-\infty}^{\infty} \psi(t) t^m \, dt = 0, \quad m = 0, \ldots, M. \quad \text{(7)}$$

The number of vanishing moments $M$ denotes the order $N$ of the Daubechies wavelet, where $N = 2M$. 

The translates of the scaling and wavelet functions on each fixed scale \( j \) form orthogonal subspaces,

\[
V_j = \{2^{j/2}\varphi(2^j t - k), \quad j \in \mathbb{Z} \},
\]

\[
W_j = \{2^{j/2}\psi(2^j t - k), \quad j \in \mathbb{Z} \},
\]

such that \( V_j \) forms a sequences of embedded subspaces

\[
\{0\}, \ldots, \subset V_{-1}, \subset V_0, \subset V_1, \ldots, \subset L^2(\mathbb{R})
\]

and

\[
V_{j+1} = V_j \oplus W_j.
\]

Let \( P_j(f)(t) \) be approximation of a function \( f(t) \) in \( L^2(\mathbb{R}) \) using \( \varphi_{j,k}(t) \) as basis, at a certain level (resolution) \( j \), then

\[
P_j(f)(t) = \sum_k c_{j,k} \varphi_{j,k}(t), \quad k \in \mathbb{Z},
\]

where \( c_{j,k} \) are the approximation coefficients. Let \( Q_j(f)(t) \) be the approximation of the function using \( \psi_{j,k}(t) \) as basis, at the same level \( j \).

\[
Q_j(f)(t) = \sum_k d_{j,k} \psi_{j,k}(t), \quad k \in \mathbb{Z},
\]

where \( d_{j,k} \) are the detail coefficients. The approximation \( P_{j+1}(f)(t) \) to the next finer level of resolution \( j + 1 \) is given by

\[
P_{j+1}(f)(t) = P_j(f)(t) + Q_j(f)(t).
\]

This forms the basis of multi-resolution analysis associated with wavelet approximation.

### 3. Reduction of wave equations to ODEs

In the isotropic Timoshenko beam, the axial and transverse motions are uncoupled. The governing differential wave equations of an isotropic Timoshenko beam, with three dofs are given as

\[
EA \frac{\partial^2 u}{\partial x^2} = \rho A \frac{\partial^2 u}{\partial t^2},
\]

\[
GAK \left[ \frac{\partial^2 v}{\partial x^2} - \frac{\partial \phi}{\partial x} \right] = \rho A \frac{\partial^2 v}{\partial t^2},
\]

\[
EI \frac{\partial^2 \phi}{\partial x^2} + GAK \left[ \frac{\partial v}{\partial x} - \phi \right] = \rho I \frac{\partial^2 \phi}{\partial t^2},
\]

where \( GAK \) and \( EI \) are the shear and bending stiffnesses, respectively, and \( \rho A, \rho I \) are the corresponding inertias. \( u(x,t), v(x,t) \) and \( \phi(x,t) \) are the axial, transverse and shear deformations respectively. Let \( u(x,t) \) be discretised at \( n \) points in the time window \([0 \ t_f]\). Let \( \tau = 0, 1, \ldots, n - 1 \)
be the sampling points, then
\[ t = \Delta t \tau, \]  
(18)

where \( \Delta t \) is the time interval between two sampling points. The function \( u(x, t) \) can be approximated by scaling function \( \phi(\tau) \) at an arbitrary scale as
\[ u(x, t) = u(x, \tau) = \sum_k u_k(x)\phi(\tau - k), \quad k \in \mathbb{Z}, \]  
(19)

where \( u_k(x) \) (referred as \( u_k \) hereafter) are the approximation coefficient at a certain spatial dimension \( x \). Similarly the other two dofs, \( v(x, t) \) and \( \phi(x, t) \) can be approximated as
\[ v(x, t) = v(x, \tau) = \sum_k v_k(x)\phi(\tau - k), \quad k \in \mathbb{Z}, \]  
(20)
\[ \phi(x, t) = \phi(x, \tau) = \sum_k \phi_k(x)\phi(\tau - k), \quad k \in \mathbb{Z}. \]  
(21)

Substituting Eqs. (18) and (19)–(21) in Eqs. (15)–(17) we get
\[ EA \sum_k \frac{d^2 u_k}{dx^2} \phi(\tau - k) = \frac{\rho A}{\Delta t^2} \sum_k u_k \phi''(\tau - k), \]  
(22)
\[ GAK \sum_k \left[ \frac{d^2 v_k}{dx^2} - \frac{d\phi_k}{dx} \right] \phi(\tau - k) = \frac{\rho A}{\Delta t^2} \sum_k v_k \phi''(\tau - k), \]  
(23)
\[ EI \sum_k \frac{d^2 \phi_k}{dx^2} \phi(\tau - k) + GAK \sum_k \left[ \frac{dv_k}{dx} - \phi_k \right] \phi(\tau - k) = \frac{\rho I}{\Delta t^2} \sum_k \phi_k \phi''(\tau - k). \]  
(24)

Taking inner product on both sides of Eqs. (22)–(23) with \( \phi(\tau - j) \), where \( j = 0, 1, \ldots, n - 1 \) we get
\[ EA \sum_k \frac{d^2 u_k}{dx^2} \int \phi(\tau - k)\phi(\tau - j) \, d\tau = \frac{\rho A}{\Delta t^2} \sum_k u_k \int \phi''(\tau - k)\phi(\tau - j) \, d\tau, \]  
(25)
\[ GAK \sum_k \left[ \frac{d^2 v_k}{dx^2} - \frac{d\phi_k}{dx} \right] \int \phi(\tau - k)\phi(\tau - j) \, d\tau = \frac{\rho A}{\Delta t^2} \sum_k v_k \int \phi''(\tau - k)\phi(\tau - j) \, d\tau, \]  
(26)
\[ EI \sum_k \frac{d^2 \phi_k}{dx^2} \int \phi(\tau - k)\phi(\tau - j) \, d\tau + GAK \sum_k \left[ \frac{dv_k}{dx} - \phi_k \right] \int \phi(\tau - k)\phi(\tau - j) \, d\tau \]  
\[ = \frac{\rho I}{\Delta t^2} \sum_k \phi_k \int \phi''(\tau - k)\phi(\tau - j) \, d\tau. \]  
(27)

The translates of scaling functions are orthogonal, i.e.
\[ \int \phi(\tau - k)\phi(\tau - j) \, d\tau = 0 \quad \text{for } j \neq k. \]  
(28)
Using Eq. (28), Eqs. (25)–(27) can be written as \( n \) simultaneous ODEs, respectively

\[
EA \frac{d^2 u_j}{dx^2} = \frac{\rho A}{\Delta t^2} \sum_{k=-N+2}^{j-N-2} \Omega_{j-k}^2 u_k, \quad j = 0, 1, \ldots, n - 1, \tag{29}
\]

\[
GAK \left[ \frac{d^2 v_j}{dx^2} - \frac{d\phi_j}{dx} \right] = \frac{\rho A}{\Delta t^2} \sum_{k=-N+2}^{j-N-2} \Omega_{j-k}^2 v_k, \quad j = 0, 1, \ldots, n - 1, \tag{30}
\]

\[
EI \frac{d^2 \phi_j}{dx^2} + GAK \left[ \frac{dv_j}{dx} - \phi_j \right] = \frac{\rho I}{\Delta t^2} \sum_{k=-N+2}^{j-N-2} \Omega_{j-k}^2 \phi_k, \quad j = 0, 1, \ldots, n - 1, \tag{31}
\]

where \( N \) is the order of the Daubechies wavelet as discussed earlier. \( \Omega_{j-k}^2 \) are the connection coefficients defined as

\[
\Omega_{j-k}^2 = \int \phi''(\tau - k)\phi(\tau - j)\,d\tau. \tag{32}
\]

Similarly, for first-order derivative \( \Omega_{j-k}^1 \) are defined as

\[
\Omega_{j-k}^1 = \int \phi'(\tau - k)\phi(\tau - j)\,d\tau. \tag{33}
\]

For compactly supported wavelets, \( \Omega_{j-k}^1, \Omega_{j-k}^2 \) are non-zero only in the interval \( k = j - N + 2 \) to \( k = j + N - 2 \). The details for evaluation of connection coefficients for different orders of derivative is given in [28].

The forced boundary condition associated with the governing differential given by Eq. (15) is

\[
EA \frac{\partial u}{\partial x} = P, \tag{34}
\]

where \( P(x, t) \) is the axial force applied. \( P(x, t) \) can be approximated similarly as \( u(x, t) \) in Eq. (19)

\[
P(x, t) = P(x, \tau) = \sum_k P_k(x)\phi(\tau - k), \quad k \in \mathbb{Z}. \tag{35}
\]

Substituting Eq. (19) and (35) in Eq. (34) and taking the inner product with \( \phi(\tau - j) \) we get

\[
EA \frac{\partial u_j}{\partial x} = P_j, \quad j = 0, 1, \ldots, n - 1. \tag{36}
\]

Similarly, corresponding to Eqs. (16) and (17), the shear force, \( V(x, t) \) and bending moment, \( M(x, t) \) are given by

\[
GAK \left[ \frac{\partial v}{\partial x} - \phi \right] = V, \tag{37}
\]

\[
EI \frac{\partial \phi}{\partial x} = M. \tag{38}
\]
Eqs. (37) and (38) can be transformed similar to Eq. (36) as

\[
GAK \left[ \frac{dv_j}{dx} - \phi_j \right] = V_j, \quad j = 0, 1, \ldots, n - 1, \tag{39}
\]

\[
EI \frac{d\phi_j}{dx} = M_j, \quad j = 0, 1, \ldots, n - 1. \tag{40}
\]

While dealing with finite length data sequence, problems arise at the boundaries. It can be observed from the ODEs given by Eq. (29) that certain coefficients \( u_j \) near the vicinity of the boundaries \( j = 0 \) and \( j = n - 1 \) lie outside the time window \([0, t_f]\) defined by \( j = 0, 1, \ldots, n - 1 \). Several approaches like capacitance matrix methods [3,29], penalty function methods for treating boundaries are reported in the literature. In this paper, first a circular convolution method is adopted assuming periodicity of the solution. By this assumption the ODEs given by Eqs. (29)–(31) are converted to a set of coupled ODEs. These ODEs are decoupled and solved using methods very similar to FSFEM. The details of the formulation is presented in the later sections. This periodic WSFE solution encounters all the problems of FSFEM in time domain analysis as discussed earlier. However, the periodic formulation allows the derivation of spectrum and dispersion relation and also the relation between the transformed ODEs in WSFEM with those in FSFEM. This leads to the direct use of WSFE for frequency domain analysis similar to FSFE.

Analysis of wave propagation in undamped finite length structures requires imposition of initial-boundary values. As stated earlier this can be done using wavelet extrapolation technique [22–24]. The effect of such treatment of boundaries on the frequency domain parameters compared to the periodic solution is obtained numerically later.

4. Periodic boundary conditions

Eqs. (29)–(30) of previous section, give \( n \) coupled ODEs each, which are to be solved for \( u_j, (v_j, \phi_j) \) using method described later. For numerical implementation, we can deal with only finite sequence. In other words, \( u(x, t) \) and hence \( u_j \) are only known in the interval \([0, t_f]\) and \( j = 0 \) to \( n \). In Eqs. (29), (30), (31) the ODEs corresponding to \( j = 0 \) to \( N - 2 \), contain coefficients \( u_j, (v_j, \phi_j) \) that lie outside the \([0, t_f]\). Similarly, on the other boundary, for \( j = (n - 1) - N + 2 \) to \( j = (n - 1) \) same problem exists.

One approach is to assume the function \( u(x, t) \), (similarly \( v(x, t) \) and \( \phi(x, t) \)) to be periodic in time, with time period \( t_f \). Thus the unknown coefficients on LHS are taken as

\[
\begin{align*}
    u_{-1} &= u_{n-1} \\
    u_{-2} &= u_{n-2} \\
    \vdots & \vdots \\
    u_{-N+2} &= u_{n-N+2}.
\end{align*}
\]

Similarly, the unknown coefficients on RHS i.e. \( u_0, u_{n+1}, \ldots, u_{n+N-2} \) are equal to \( u_0, u_1, \ldots, u_{N-2} \), respectively. With the above assumption, the coupled ODEs given by Eqs. (29) can be written in
matrix form as
\[
\begin{pmatrix}
\frac{d^2 u_j}{dx^2}
\end{pmatrix} = \frac{\rho A}{EA} A^2 \{u_j\},
\]
where \( A^2 \) are \( n \times n \) circulant connection coefficient matrices and have the form
\[
A^2 = \frac{1}{\Delta t^2} \begin{bmatrix}
\Omega^2_0 & \Omega^2_{-1} & \cdots & \Omega^2_{N+2} & \cdots & \Omega^2_{N-2} & \cdots & \Omega^2_{1} \\
\Omega^2_{1} & \Omega^2_0 & \cdots & \Omega^2_{N+3} & \cdots & 0 & \cdots & \Omega^2_{2} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\Omega^2_{-1} & \Omega^2_{-2} & \cdots & 0 & \cdots & \Omega^2_{N-3} & \cdots & \Omega^2_{0}
\end{bmatrix},
\]
(43)
\[A^1\] for first-order derivative required to introduce damping has a similar form. For a circulant matrices \( A^1 \) (and \( A^2 \)) [30], the eigenvalues \( \lambda_j \) are
\[
\lambda_j = \frac{1}{\Delta t} \sum_{k=-N+2}^{N-2} \Omega_k^1 e^{-2\pi ijk/n}, \quad j = 0, 1, \ldots, n - 1
\]
(44)
where \( i = \sqrt{-1} \) and the corresponding orthonormal eigenvector \( \omega_j, j = 0, 1, \ldots, n - 1 \) are
\[
(\omega_j)_k = \frac{1}{\sqrt{n}} e^{-2\pi ijk/n}, \quad k = 0, 1, \ldots, n - 1.
\]
(45)
For \( A^1, \Omega^1_p = -\Omega^1_{-p} \) for \( p = 1, 2, \ldots, N - 2 \) and \( \Omega^1_0 = 0 \) and we can write \( \lambda_j = i\beta_j \) where
\[
\beta_j = -\frac{2}{\Delta t} \sum_{k=1}^{N-2} \Omega^1_k \sin \left[ \frac{2\pi kj}{n} \right], \quad j = 0, 1, \ldots, n - 1.
\]
(46)
It can be seen from the above derivations that the wavelet coefficients of first and second derivatives can be obtained as
\[
\{\dot{u}_j\} = A^1 \{u_j\},
\]
(47)
\[
\{\ddot{u}_j\} = A^2 \{u_j\}.
\]
(48)
The second derivative can also be written as
\[
\{\ddot{u}_j\} = A^1 \{\dot{u}_j\}.
\]
(49)
Substituting Eq. (47) in Eq. (49) we get
\[
\{\ddot{u}_j\} = [A^1]^2 \{u_j\}.
\]
(50)
Thus though the second-order connection coefficient matrices \( A^2 \) can be evaluated independently [28], they can also written as
\[
\]
(51)
The above modification is done as this form helps in imposing the initial conditions for non-periodic solution discussed later. Thus the Eq. (42) can be written as

\[
\frac{d^2 u_j}{dx^2} = \frac{\rho A}{EA} [A']^2 \{u_j\}.
\] (52)

As discussed earlier, the spectral element formulation in the later part of the paper, involves eigenvalue analysis. This is done to diagonalise the matrix in Eq. (42) and decouple the ODEs. For periodic boundary condition, these eigenvalues are known analytically and hence decreases the computational cost. Thus, the matrix \( A' \) can be written as

\[
A' = \Phi \Pi \Phi^{-1},
\] (53)

where \( \Pi \) is the diagonal matrix containing the diagonal terms \( i \beta_j \) and \( \Phi \) is the eigenvector matrix. Using Eq. (53), Eq. (42) can be decoupled and written as

\[
\frac{d^2 \hat{u}_j}{dx^2} = -\frac{\rho A}{EA} \beta_j^2 \hat{u}_j, \quad j = 0, 1, \ldots, n - 1,
\] (54)

where

\[
\hat{u}_j = \Phi^{-1} u_j.
\] (55)

Eqs. (30) and (31) can be written similarly as

\[
GAK \left[ \frac{d^2 \hat{v}_j}{dx^2} - \frac{d \hat{\phi}_j}{dx} \right] = -\rho A \beta_j^2 \hat{v}_j, \quad j = 0, 1, \ldots, n - 1,
\] (56)

\[
EI \frac{d^2 \hat{\phi}_j}{dx^2} + GAK \left[ \frac{d \hat{v}_j}{dx} - \hat{\phi}_j \right] = -\rho I \beta_j^2 \hat{\phi}_j, \quad j = 0, 1, \ldots, n - 1.
\] (57)

5. Frequency domain analysis

For periodic solution, the wavelet transformation given by Eq. (19) (or (20), (21)) can be written as matrix equation [23]

\[
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_{n-1}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & \ldots & \varphi_{N-2} & \ldots & \varphi_2 & \varphi_1 \\
\varphi_1 & 0 & 0 & \ldots & 0 & \ldots & \varphi_3 & \varphi_2 \\
\varphi_2 & \varphi_1 & 0 & \ldots & 0 & \ldots & \varphi_4 & \varphi_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\varphi_{N-2} & \varphi_{N-3} & \varphi_{N-4} & \ldots & \ldots & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \varphi_{N-3} & \varphi_2 & \varphi_1 & 0
\end{bmatrix}
\begin{bmatrix}
u_0 \\
u_1 \\
u_2 \\
\vdots \\
u_{n-1}
\end{bmatrix}.
\] (58)
where \( U_j, \varphi_j \) are the values of \( u(x, \tau) \) and \( \varphi(\tau) \) at \( \tau = j \). For such circulant matrix the Eq. (58) can be replaced by a convolution relation which can be written as

\[
\{ \tilde{U}_j \} = \{ \tilde{K}_{\varphi j} \tilde{u}_j \},
\]

(59)

where \( \{ \tilde{U}_j \}, \{ \tilde{u}_j \} \) are FFT of \( \{ U_j \} \) and \( \{ u_j \} \), respectively. \( \{ \tilde{K}_{\varphi j} \} \) is FFT of \( \{ K_{\varphi} \} = [0 \ \varphi_1 \ \varphi_2 \ \ldots \ \varphi_{N-2} \ \ldots \ 0] \) which is the first column of the scaling function matrix given in Eq. (58). Similarly, in Eq. (52) the matrix \( \Lambda^1 \) is also a circulant matrix and thus it can be written as

\[
\left\{ \frac{d^2 \tilde{u}_j}{dx^2} \right\} = \frac{\rho A}{EA} \{ \tilde{K}_{\varphi j}^2 \tilde{u}_j \},
\]

(60)

where \( \{ \tilde{K}_{\varphi j} \} \) are the FFT of \( K_{\varphi} = [\varphi_0^1 \ \varphi_{-1}^1 \ \ldots \ \varphi_{-N+2}^1 \ \ldots \ \varphi_{N-2}^1 \ \ldots \ \varphi_1^1] \) which is the first column of the connection coefficient matrix \( \Lambda^1 \). Substituting Eq. (59) in Eq. (60) we get

\[
\left\{ \frac{d^2 (\tilde{U}_j/\tilde{K}_{\varphi j})}{dx^2} \right\} = \frac{\rho A}{EA} \{ \tilde{K}_{\varphi j}^2 (\tilde{U}_j/\tilde{K}_{\varphi j}) \}
\]

(61)

or

\[
\left\{ \frac{d^2 \tilde{U}_j}{dx^2} \right\} = \frac{\rho A}{EA} \{ \tilde{K}_{\varphi j}^2 \tilde{U}_j \}.
\]

(62)

It can be easily seen that the FFT coefficients \( \tilde{K}_{\varphi j} \) are equal to the eigenvalues \( i\beta_j \) of the matrix \( \Lambda^1 \) given by Eq. (44). Thus Eq. (62) can be written as

\[
\frac{d^2 \tilde{U}_j}{dx^2} = -\frac{\rho A}{EA} \beta_j^2 \tilde{U}_j, \quad j = 0, 1, \ldots, n - 1.
\]

(63)

It should be mentioned here that, by relating the Eqs. (54) and (63), it can be observed that the transformation given by Eq. (55) is similar to FFT for periodic WSFE formulation.

In FSFEM, the transformed ODEs are of the form

\[
\frac{d^2 \tilde{U}_j}{dx^2} = -\frac{\rho A}{EA} \omega_j^2 \tilde{U}_j, \quad j = 0, 1, \ldots, n - 1,
\]

(64)

where

\[
\omega_j = \frac{2\pi j}{n\Delta t}.
\]

(65)

It can be seen that for a given sampling rate \( \Delta t \), \( \beta_j \) exactly matches \( \omega_j \) up to a certain fraction of Nyquist frequency \( f_{nyq} = 1/2\Delta t \). Thus similar to FSFE, WSFE can be used directly for studying frequency-dependent characteristics like spectrum and dispersion relation but up to a certain fraction of \( f_{nyq} \). This fraction is dependent on the order of basis and is more for higher-order
In Figs. 1(a), $\omega_j$, $\beta_j$ and $\gamma_j$ are compared with respect to fraction of $f_{nyq}$. This study also helps to determine the sampling rate required depending on the frequency content of excitation loads and order of bases. This has been explained with numerical experiments later.
6. Non-periodic boundary condition

For non-periodic solution the boundaries are treated using wavelet extrapolation method for Daubechies compactly supported wavelets. The detail of the formulation is given in [22–24]. In brief this method uses a polynomial of order \( p - 1 \) \((p = N/2)\) to extrapolate the values at the boundaries. Since, in this work the wavelets are used in time, the unknown coefficients on the LHS (i.e. \( u_{-1}, u_{-2}, \ldots, u_{N+2} \)) are extrapolated from the initial values. The coefficients \( u_n, u_{n+1}, \ldots, u_{n+N-2} \) on RHS are extrapolated from the known coefficients \( u_{(n-1)-p+1}, u_{(n-1)-p+2}, \ldots, u_{n-1} \).

As discussed in the earlier work by authors [21], after treating the boundaries, the ODEs given by Eq. (29) can be written in matrix form, similar to Eq. (52)

\[
\begin{bmatrix}
\frac{d^2 u_j}{dx^2}
\end{bmatrix} = \frac{\rho A}{EA} [I^1]^2 \{u_j\},
\]

where \( I^1 \) is the first-order connection coefficient matrix. These coupled ODEs are similarly decoupled using eigenvalue analysis

\[
I^1 = \Phi \Pi \Phi^{-1},
\]

where \( \Pi \) is the diagonal eigenvalue matrix and \( \Phi \) is the eigenvectors matrix of \( I^1 \). Let the eigenvalues be \( i_j \), then the decoupled ODEs corresponding to Eqs. (54), (56) and (57) can be written as

\[
\frac{d^2 \hat{u}_j}{dx^2} = -\frac{\rho A}{EA} i_j^2 \hat{u}_j, \quad j = 0, 1, \ldots, n - 1,
\]

\[
GAK \left[ \frac{d^2 \hat{v}_j}{dx^2} - \frac{d \hat{\phi}_j}{dx} \right] = -\rho A i_j^2 \hat{v}_j, \quad j = 0, 1, \ldots, n - 1,
\]

\[
EI \frac{d^2 \hat{\phi}_j}{dx^2} + GAK \left[ \frac{d \hat{v}_j}{dx} - \hat{\phi}_j \right] = -\rho I_i^2 \hat{\phi}_j, \quad j = 0, 1, \ldots, n - 1.
\]

The spectral element formulation from these decoupled ODEs (Eqs. (68)-(70)) is similar as that for periodic solution except that, \( \beta_j \) is replaced by \( \gamma_j \).

Unlike \( \beta_j \) which is real, \( \gamma_j \) is complex. However, from numerical experiments it is seen that the real part of \( \gamma_j \) matches \( \beta_j \) which are compared for different order of basis in Fig. 1(a). The additional imaginary part of \( \gamma_j \) are plotted for different basis in Fig. 1(b).

7. Spectral element formulation

From the previous sections, we get a set of ODEs (Eqs. (54), (56), (57)) for Timoshenko beam with axial, transverse and shear modes, in a transformed wavelet domain. These equations are required to be solved for \( \hat{u}_j, \hat{v}_j, \hat{\phi}_j \) and the actual solutions \( u(x, t), v(x, t), \phi(x, t) \) are obtained using inverse wavelet transform. For finite length data, the wavelet transform and its inverse can be obtained using a transformation matrix [31]. For periodic solution the transformation matrix is
given by Eq. (58). For non-periodic solution, required for finite length structures, the transformation matrix needs to be modified using wavelet extrapolation technique and this has been described in previous section.

It can be seen that the transformed ODEs have a form which is similar to those in FSFEM [9]. Thus WSFE can be formulated following the same method as for FSFE formulation and only \( \omega_j \) are replaced by \( \beta_j \). In this section, the subscript \( j \) is dropped hereafter for simplified notations and all the following equations are valid for \( j = 0, 1, \ldots, n - 1 \).

The exact interpolating functions for an element of length \( L \), obtained by solving Eqs. (54), (56) and (57), respectively, are

\[
\hat{u}(x) = Ae^{-ik_1x} + Be^{-ik_1(L-x)},
\]
\[
\hat{v}(x) = Ce^{-ik_2x} + De^{-ik_3x} + Ec^{-ik_2(L-x)} + Fe^{-ik_3(L-x)},
\]
\[
\hat{\phi}(x) = P_1Ce^{-ik_2x} + P_2De^{-ik_3x} + P_3Ec^{-ik_2(L-x)} + P_4Fe^{-ik_3(L-x)}.
\]

\( \{P_1, P_2, P_3, P_4\} \) are the amplitude ratios for each set of \( k_1, k_2 \) and \( k_3 \) and \( k_1, k_2 \) and \( k_3 \) are obtained by substituting Eqs. (71)–(72) into Eqs. (54), (56) and (57). \( k_1, k_2 \) and \( k_3 \) corresponds to the three modes i.e. axial, transverse and shear, respectively. As discussed earlier, \( \beta \) matches exactly with \( \omega \) up to a certain fraction of \( \omega_{nyq} \) and hence \( k_1, k_2, k_3 \) will match the wavenumbers \( (k_{f_1}, k_{f_2}, k_{f_3}) \) in FSFEM. Therefore these \( k \) in WSFEM can be used for obtaining the spectrum and dispersion relations. This is explained in detail with numerical examples in the later section. \( k_1, k_2 \) and \( k_3 \) can be obtained as

\[
k_1 = \frac{\beta}{c_0},
\]
\[
k_2, k_3 = \left\lfloor \frac{1}{2}\left\{ \left( \frac{1}{c_s} \right)^2 + \left( \frac{Q}{c_{0q}} \right)^2 \right\} \beta^2 \pm \sqrt{\left( \frac{\beta}{c_{0q}} \right)^2 + \frac{1}{4} \left\{ \left( \frac{1}{c_{0q}} \right)^2 - \left( \frac{Q}{c_{0q}} \right)^2 \right\} \beta^4} \right\rfloor^{1/2},
\]

where the constants \( c_0 = \sqrt{EA/\rho A}, c_{0q} = \sqrt{EI/\rho A}, c_s = \sqrt{GAK/\rho A} \) and \( Q = \sqrt{\rho J/\rho A} \). The Euler–Bernoulli beam solution can be obtained by setting \( Q = 0 \) and \( GAK = \infty \).

\( \{a = A, B, C, D, E, F\} \) are the unknown coefficients to be determined from transformed nodal displacements \( \{\hat{u}\} \). \( \{\hat{u}\} = \{\hat{u}_1 \hat{v}_1 \hat{\phi}_1 \hat{u}_2 \hat{v}_2 \hat{\phi}_2\} \) where, \( \hat{u}_1 \equiv \hat{u}(0), \hat{v}_1 \equiv \hat{v}(0), \hat{\phi}_1 \equiv \hat{\phi}(0) \) and \( \hat{u}_2 \equiv \hat{u}(L), \hat{v}_2 \equiv \hat{v}(L), \hat{\phi}_2 \equiv \hat{\phi}(L) \) (see Fig. 2 for the details of dof the element can support). Thus, we can relate

![Fig. 2. Timoshenko beam element with nodal displacements and forces.](image-url)
the nodal displacements and unknown coefficients as
\[
\{\hat{u}\} = [B]\{a\}.
\] (76)

From the forced boundary conditions, (Eqs. (36), (39) and (40)), nodal forces and unknown coefficients can be related as
\[
\{\hat{F}\} = [C]\{a\},
\] (77)
where \(\{\hat{F}\} = \{\hat{P}_1 \hat{V}_1 \hat{M}_1 \hat{P}_2 \hat{V}_2 \hat{M}_2\}\) and \(\hat{P}_1 = \hat{P}(0), \hat{V}_1 = \hat{V}(0), \hat{M}_1 = \hat{M}(0)\) and \(\hat{P}_2 = \hat{P}(L), \hat{V}_2 = \hat{V}(L), \hat{M}_2 = \hat{M}(L)\) (see Fig. 2). From Eqs. (76) and (77) we can obtain a relation between transformed nodal forces and displacements similar to conventional FE
\[
\{\hat{F}\} = [C][B]^{-1}\{\hat{u}\} = [\hat{K}][\hat{u}],
\] (78)
where \([\hat{K}]\) is the dynamic stiffness matrix. After the constants \(\{a\}\) are known from the above equations, they can substituted back to Eqs. (71)–(73) to obtain the transformed displacements \(\hat{u}, \hat{v}, \hat{\phi}\) at any given \(x\).

From the correspondence established between the transformed ODEs in FSFEM and WSFEM in the previous section, the forward and backward moving waves can be identified in WSFEM. Thus, a semi-infinite spectral element with single node can be formulated in WSFEM similar to FSFEM, particularly for periodic solution. In FSFEM, throw-off or semi-infinite elements are normally used to allow leakage of response so that the signal does not wrap around. The use of semi-infinite element in wavelet analysis is minimal as the method can efficiently handle finite structure without any such problems by the use of non-periodic WSFEM solution.

8. Numerical experiments

Here, first WSFEM is used for frequency and time domain analysis of wave propagation in an elementary rod. Though non-periodic WSFEM solution is free from the problems associated with the time domain analysis using FSFEM, FSFEM can be used effectively to study frequency-dependent wave parameters. As discussed in Section 5, WSFEM can be used to obtain the frequency-dependent wave characteristics similar to FSFEM, but up to a fraction of Nyquist frequency \(f_{nyq}\). Numerical experiments are performed to study the spectrum and dispersion relations using WSFEM and also the effect of non-periodic WSFE formulation on these parameters. In addition, emphasis is given to numerically evaluate the required sampling rate \(\Delta t\) for accurate analysis with the proposed WSFEM depending on the excitation frequency and order of the basis. The above derivation is done by comparison of frequency domain characteristics obtained using WSFEM with those obtained using FSFEM. Further, using larger \(\Delta t\) than that is mandated leads to spurious dispersion modes of propagation in time domain analysis and this have been demonstrated with several examples. Both modulated sinusoidal pulse and broadband impulse loads are used as excitation in these examples. For studying the response due to modulated pulse, an infinite rod is considered and thus periodic WSFEM is used, while for impulse loading, finite length rod and non-periodic WSFE solution is considered. Next, similar experiments are presented for Euler–Bernoulli beam. Finally, such analysis is done for Timoshenko beam which has two coupled propagating modes namely flexure and shear. This
example is considered as the shear mode starts propagating only after a certain cut-off frequency. Thus, the frequency content of the modulated pulse should be higher to capture this mode unlike in elementary rod and Euler–Bernoulli beam.

All the numerical examples presented in this section are for an isotropic aluminum rod and beam of width \( (= 2b) \) 6 mm and depth \( (= 2b) \) 25 mm. The elastic properties are as follows, Young’s modulus \( E = 70 \text{ GPa} \), shear modulus \( G = 27 \text{ GPa} \) and density \( \rho = 2700 \text{ kg/m}^3 \). The shear correction factor \( K = 0.85 \). As mentioned earlier, the Euler–Bernoulli solutions are obtained considering \( GAK = \infty \) and \( \rho I = 0 \).

### 8.1. Elementary rod

The spectrum relation for an elementary rod is obtained using WSFEM with \( \Delta t = 1 \mu s \) \( (f_{\text{nyq}} = 500 \text{ kHz}) \) and \( \Delta t = 2 \mu s \) \( (f_{\text{nyq}} = 250 \text{ kHz}) \) and are presented in Figs. 3(a) and (b), respectively. The comparison with FSFEM results in each case, shows that the non-dimensional wavenumber \( k_1h \) obtained using periodic WSFEM is exact up to a fraction \( p_N \) of \( f_{\text{nyq}} \) beyond which spurious dispersion is observed. It can be seen from Figs. 3(a) and (b) that the above mentioned fraction \( p_N \) varies only with order of basis \( N \) and is independent of the problem. For \( N = 22 \), \( p_N \approx 0.6 \), while for \( N = 6 \), \( p_N \approx 0.36 \). Thus, for \( \Delta t = 1 \mu s \), wave characteristics can be obtained for a frequency range of \( f_{\text{nyq}} = 500 \text{ kHz} \) using FSFEM and for a frequency range of \( f_N = 0.6f_{\text{nyq}} = 300 \text{ kHz} \) using WSFEM with \( N = 22 \).

In Figs. 4(a) and (b), the spectrum relation derived from non-periodic WSFEM with \( N = 22 \) and 6 and FSFEM are presented for \( \Delta t = 4 \mu s \) and \( \Delta t = 8 \mu s \), respectively. For analysis of finite length structures non-periodic WSFE formulation is adopted as it allows imposition of boundary condition and hence remove wrap around problem. As discussed in earlier sections and shown in Fig. 1(b), non-periodic formulation adds an imaginary part to the real wavenumber \( k_1 \) obtained from periodic WSFEM. This imaginary part is presented in Fig. 4(c) for \( N = 22 \) and 6 obtained with \( \Delta t = 4 \mu s \). This imaginary part can be physically thought of damping.

Next, the dispersion relation for rod is plotted. In rod the waves are non-dispersive and hence the group speed and phase speeds are same i.e. the ratio of \( C_g/C_0 = 1 \). In Fig. 5, the dispersion relation derived from WSFEM with \( N = 22 \) is plotted for \( \Delta t = 2 \mu s \) or \( f_{\text{nyq}} = 250 \text{ kHz} \) and compared to that obtained from FSFEM. In this figure the non-dimensional group speed \( C_g/C_0 \) where \( C_g = \text{real}(de_0/dk_1) \) and \( C_0 = \sqrt{EA/\rho A} \) are plotted with respect to frequency. Similar to spectrum relation, WSFEM predicts the exact speeds up to almost \( f_N \) and is approximately equal to 150 kHz here for \( N = 22 \).

As mentioned earlier, a consequence of the above study is a priori determination of \( \Delta t \) required by WSFEM depending on the excitation frequency and order of basis \( N \). This is explained in detail through numerical experiments. First, the response to sinusoidal pulse modulated at 110 and 200 kHz, simulated using periodic WSFEM is presented. For such loading, the waves propagate non-dispersively. The pulse spectrum are superimposed in Figs. 3(a) and (b). The loads are applied at a point \( C \) (see Fig. 6) on an infinite beam in axial direction and the axial velocity is measured at \( D \) at \( L = 0.5 \text{ m} \) away from \( C \). In Fig. 7(a), the axial velocities obtained using WSFEM \( (N = 22 \text{ and } 6) \) and FSFEM with \( \Delta t = 1 \mu s \) for sinusoidal pulse at 200 kHz are presented. It can be seen that in this case, the result obtained using WSFEM with \( N = 22 \) matches exactly with FSFEM, while for \( N = 6 \), it varies considerably. This observation can be explained using.
Fig. 3. Spectrum relation for elementary rod $k_1 h$ (and pulse spectrum ‘—.’) for sampling rate (a) $\Delta t = 1 \mu s$ and (b) $\Delta t = 2 \mu s$. 
Fig. 4. Spectrum relation for elementary rod $k_1 h$ (and impulse spectrum ‘– –’ ) for sampling rate (a) $\Delta t = 4 \mu s$ (b) $\Delta t = 8 \mu s$ and (c) imaginary part of $k_1 h$ introduced due to non-periodic WSFEM solution with $\Delta t = 4 \mu s$. 
Fig. 3(a). It can be seen from Fig. 3(a) that in the frequency range of the applied sinusoidal pulse of 200 kHz, WSFEM with \( N = 22 \) predicts exact wavenumbers but WSFEM with \( N = 6 \) fails to do so. Thus the results simulated with \( N = 6 \) in Fig. 7(a) cannot capture the correct wave propagation. In Fig. 7(b) similar axial velocities are presented but for excitation with sinusoidal pulse modulated at 110 kHz. In this case, both the results obtained using WSFEM with \( N = 22 \) and 6 match exactly with FSFEM results. This is expected as it can be seen from Fig. 3(a) that for the frequency range of the 110 kHz loading, WSFEM with \( N = 22 \) and 6 give the correct spectrum relation. From these experiments it can be summarised that the time domain results simulated
with WSFEM will be accurate if the frequency content of the load is within the range where WSFEM predict the actual spectrum relation. As this allowable range for a given $N$ (denoted as $f_N$ hereafter) is a fixed fraction $p_N$ of $f_{nyq}$, it can be increased or decreased by decreasing or

\[ p_N = \frac{f_N}{f_{nyq}} \]

Fig. 7. Axial velocity measured at D (Fig. 6), due to sinusoidal pulse modulated at (a) 200 kHz and (b) 110 kHz, applied at C with $\Delta t = 1 \mu s$ and $L = 0.5$ m.
increasing $\Delta t$, respectively. In Fig. 8(a) and (b), results similar to Fig. 7(a) and (b), respectively are presented but here the simulations are done with $\Delta t = 2 \mu$s and $L = 0.5\,\text{m}$.

Fig. 8. Axial velocity measured at D (Fig. 6), due to sinusoidal pulse modulated at (a) 200 kHz and (b) 110 kHz, applied at C with $\Delta t = 2 \mu$s and $L = 0.5\,\text{m}$. By increasing $\Delta t$, $f_{\text{nyq}}$ and correspondingly $f_N$ are decreased to 250 and 150 kHz ($N = 22$), 90 kHz ($N = 6$), respectively. As
for $\Delta t = 2 \mu s$, $f_N$ is less than 200 kHz for both $N = 22$ and 6, this sampling rate cannot simulate the wave propagation due to 200 kHz modulated sinusoidal loading as shown in Fig. 8(a) whereas FSFEM gives correct results as $f \text{_{nyq}}$ is greater than 200 kHz. Similarly, for sinusoidal loading of 110 kHz, though WSFE solution with $N = 22$ matches exactly with FSFEM result as shown in Fig. 8(b), WSFEM with $N = 6$ is unable to capture the actual response. This is justified as it can be interpreted from Fig. 3(b), that for $N = 22$ the excitation frequency lies within the range $f_N = 150$ kHz and is otherwise for $N = 6$, where $f_N = 90$ kHz. Thus for WSFEM with given $N$, $\Delta t$ or $f \text{_{nyq}}$ should be such that $f_N$ is greater than the excitation frequency.

Next, non-periodic WSFEM is used for analysis of wave propagation in an undamped fixed-free rod of length $L = 0.25$ m. An unit broadband impulse load is applied at the free end in axial direction and axial velocity is measured at the tip. The load has a duration of 50 $\mu$s and frequency content of 44 kHz. As the excitation frequency here is much lower than that for modulated sinusoidal pulse used in previous examples, higher $\Delta t$ of 4 $\mu$s ($f \text{_{nyq}} = 128$ kHz) and 8 $\mu$s ($f \text{_{nyq}} = 64$ kHz) are considered. The impulse spectrum is superimposed in Figs. 4(a) and (b). For $\Delta t = 4 \mu$s, all the frequencies i.e. $f \text{_{nyq}} = 128$ kHz, $f_N \approx 77$ kHz for $N = 22$ and $f_N \approx 46$ kHz for $N = 6$, are higher than the excitation frequency and this can also be interpreted from Fig. 4(a). Thus, it is expected that wave propagation analysis with WSFEM for $\Delta t = 4 \mu$s will be correct for both $N = 22$ and 6. Similarly, as shown in Fig. 4(b), $f \text{_{nyq}} = 64$ kHz, $f_N \approx 38$ kHz for $N = 22$ and $f_N \approx 23$ kHz for $N = 6$. In this case the excitation frequency band exceed $f_N$ for $N = 6$ and also slightly for $N = 22$ and thus unlike the previous case, this will not simulate the exact response. These predictions are validated with the responses presented in Figs. 9(a) and (b). In Fig. 9(a), the tip axial velocity due to tip impulse load obtained using WSFEM ($N = 22$ and 6) and $\Delta t = 4 \mu$s are plotted and compared with FSFEM solution. Both WSFEM and FSFEM require a single element to give the above results which validates the exactness of the formulated dynamic stiffness matrix. However, the FSFEM solution is obtained using an additional single-noded throw-off element of stiffness 100 times that of the actual element to reduce wrap around problem. The time window $T_w$ required by FSFEM is 32 768 $\mu$s to completely remove the distortions due to wrap around. However, WSFE solution is obtained with $T_w = 512$ $\mu$s as the accuracy of such solution is independent of $T_w$. It can be seen that for $\Delta t = 4 \mu$s WSFE solution matches exactly with FSFEM solution for both $N = 22$ and $N = 6$ as expected. In Fig. 9(b), similar results are plotted except that $\Delta t = 8 \mu$s here. It can be observed that WSFE solution for $N = 6$ is highly distorted in this case while for $N = 22$ the response matches quite well with FSFE solution except of small deviation. This can be justified as $f_N$ for $N = 22$ is very close to the excitation frequency but is otherwise for $N = 6$. Thus, $\Delta t$ required for accurate simulation of wave propagation in finite structure due to broadband impulse loading with non-periodic WSFEM has to be determined similarly as for the previous cases of modulated sinusoidal excitations.

All the frequency and time domain analysis presented so far is done for $N = 22$ and 6. As observed from the previous experiments, the fraction $p_N$ corresponding to allowable frequency range $f_N$ is dependent only on order of Daubechies scaling function basis $N$. In Fig. 10(a), the percentage error in form of $\frac{|k_r-k_f|}{k_r}$ is plotted with respect to the fraction $f_j/f \text{_{nyq}}$ for different order of basis functions where, $k_r$, $k_f$ are wavenumbers obtained using WSFEM and FSFEM, respectively. It can be seen that the calculated error is negligible and almost equal to zero up to a certain $f_j/f \text{_{nyq}}$ and then increases quite steeply. For a given $N$, the fraction $p_N$ as defined earlier is
the highest frequency fraction up to which the error remains negligible and this can be derived numerically from Fig. 10(a) for different $N$. Though, here the wavenumbers correspond to an elementary rod, the same relation will hold for wavenumbers of other waveguides.
In Fig. 10(b), the refinement achieved in time domain analysis of wave propagation by gradually decreasing $\Delta t$ is plotted for different $N$. The tip axial velocity in the fixed-free rod due to tip impulse load for $N = 22$ and $\Delta t = 1 \mu s$ is taken to be the most refined solution $v_f$. The time
window is $T_w = 512\,\mu\text{s}$. The coarser solutions $v_c$ are obtained by increasing $\Delta t$ from 2 to 8 $\mu\text{s}$ keeping $T_w = 512\,\mu\text{s}$ fixed and thus decreasing the number of sampling points $n$. The error is calculated $\frac{|v_f - v_c|}{\sqrt{n}}$ for different $\Delta t$ and are plotted for different $N$. As discussed earlier, the simulation using WSFEM will become erroneous when the excitation frequencies are more than the allowable frequency range $f_N$. It can be seen from Fig. 10(b), that for a given $\Delta t$, the error gradually decreases with increasing $N$. This is due to the fact that with higher $N$, $p_N$ and thus $f_N$ increases and the excitation frequency i.e. 44 kHz, becomes close to $f_N$. Hence the accuracy of the simulated response for a given $\Delta t$ increases with $N$.

8.2. Euler–Bernoulli beam

In Fig. 11, the spectrum relation for an Euler–Bernoulli beam obtained using periodic WSFEM is presented and compared with those derived from FSFEM. Similar to elementary rod, even here, the wavenumber derived from WSFEM predict the exact wave behaviour up to the fraction $p_N$ of $f_{\text{nyq}}$ beyond which it shows spurious dispersions. As mentioned earlier, $p_N$ is independent of problem and is fixed for given $N$. The value of $p_N$ can be derived numerically from Fig. 10(a) for different $N$. In Fig. 11 the non-dimensional wavenumbers $k_2h$ and $k_3h$ obtained with $\Delta t = 1\,\mu\text{s}$ or $f_{\text{nyq}} = 500\,\text{kHz}$ are plotted. In Fig. 12(a) the non-dimensional wavenumbers $k_2h$ and $k_3h$ derived from non-periodic WSFEM for $\Delta t = 4\,\mu\text{s}$ are plotted. In Fig. 12(b), the imaginary and real part of $k_2h$ and $k_3h$, respectively, that are introduced due to the imposition of boundary conditions are plotted for $\Delta t = 4\,\mu\text{s}$.

![Figure 11](image-url)

Fig. 11. Spectrum relation for Euler–Bernoulli beam $k_2h$ and $k_3h$ (and pulse spectrum ‘—.’) for sampling rate $\Delta t = 1\,\mu\text{s}$.
Fig. 12. (a) Spectrum relation for Euler–Bernoulli beam $k_2h$ and $k_3h$ (and impulse spectrum ‘--‘) for sampling rate $\Delta t = 4\mu s$ and (b) imaginary part of $k_2h$ and real part of $k_3h$ introduced due to non-periodic WSFEM solution with $\Delta t = 4\mu s$. 

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**Fig. 12. (a) Spectrum relation for Euler–Bernoulli beam $k_2h$ and $k_3h$ (and impulse spectrum ‘--‘) for sampling rate $\Delta t = 4\mu s$ and (b) imaginary part of $k_2h$ and real part of $k_3h$ introduced due to non-periodic WSFEM solution with $\Delta t = 4\mu s$.**
In Fig. 13, the dispersion relation of Euler–Bernoulli beam derived from WSFEM with \( N = 22 \) is plotted for \( \Delta t = 2 \mu s \) or \( f_{\text{nyq}} = 250 \text{kHz} \) and compared to that obtained from FSFEM. In this figure the non-dimensional group speed \( C_g/C_0 \) where \( C_g = \text{real}(\omega_0/dk) \) and \( C_0 = \sqrt{EA/\rho A} \) are plotted with respect to frequency. Similar to spectrum relation, WSFEM predicts the exact speeds up to almost \( f_N \) and is approximately equal to \( 150 \text{kHz} \) here for \( N = 22 \).

In Fig. 13, the dispersion relation of Euler–Bernoulli beam derived from WSFEM with \( N = 22 \) is plotted for \( \Delta t = 2 \mu s \) or \( f_{\text{nyq}} = 250 \text{kHz} \) and compared to that obtained from FSFEM. In this figure the non-dimensional group speed \( C_g/C_0 \) where \( C_g = \text{real}(\omega_0/dk) \) and \( C_0 = \sqrt{EA/\rho A} \) are plotted with respect to frequency. Similar to spectrum relation, WSFEM predicts the exact speeds up to almost \( f_N \) and is approximately equal to \( 150 \text{kHz} \) here for \( N = 22 \).

Fig. 14 shows the transverse velocity of non-dispersive flexural wave due to sinusoidal pulse modulated at \( 200 \text{kHz} \) applied in transverse direction. The load is applied at a point \( C \) (see Fig. 6) on an infinite beam and the velocity is measured at \( D \) at a distance \( L = 0.5 \text{m} \) from \( C \). In Fig. 14, the response obtained with WSFEM using \( \Delta t = 2 \mu s, N = 22 \) and 6 are compared with FSFEM results. The results for \( N = 22 \) matches exactly with FSFEM results whereas for \( N = 6 \) slight deviations are observed. The explanation for the above observations are similar to those given for elementary rod and is as follows. The spectrum of the pulse is plotted in Fig. 11 and it can be seen that for \( \Delta t = 1 \mu s \) the excitation frequency i.e. \( 200 \text{kHz} \) is slightly more than \( f_N \) for \( N = 6 \) but less for \( N = 22 \).

Next, WSFE is used to study wave propagation in a cantilever beam with \( L = 0.25 \text{m} \) due to transverse unit impulse load applied at the free end. As for rod, non-periodic WSFEM is used here, for such finite length beam. As the excitation frequency (44 kHz) is lower here, higher \( \Delta t \) can be used and the impulse load spectrum is superimposed in spectrum relation shown in Fig. 12(a) with \( \Delta t = 4 \mu s \). For this \( \Delta t \), since the excitation frequency is within the range \( f_N \) for both \( N = 22 \) and 6, WSFEM will be able to give the exact results. In Fig. 15, this tip transverse velocity is plotted for WSFEM with \( N = 22, 6 \) and FSFEM, respectively. For comparison with FSFEM a damping of \( \eta = 0.5 \) is considered in all the cases. While FSFEM requires a time window \( T_w = \)
Fig. 14. Transverse velocity measured at D (Fig. 6), due to sinusoidal pulse modulated at 200 kHz applied at C with $\Delta t = 1 \mu s$ and $L = 0.5 \text{m}$.

Fig. 15. Transverse tip velocity of a fixed-free Euler–Bernoulli beam ($L = 0.25 \text{m}$) due to tip unit impulse load applied in transverse direction with $\Delta t = 4 \mu s$. 
Fig. 16. Spectrum relation for Timoshenko beam $k_2 h$ and $k_3 h$ using (a) $\Delta t = 2 \mu s$ (and pulse spectrum ‘--’ ) and (b) $\Delta t = 8 \mu s$ (and impulse spectrum ‘--’ ).
32.768 μs to remove the distortions due to wrap around, WSFEM use $T_w = 512$ μs. It can be seen that all the plots compare very well. For both WSFEM and FSFEM solutions, a single element is used as they are capable of capturing the exact mass distribution.

8.3. Timoshenko beam

Timoshenko beam has two coupled mode namely bending and shear modes. In Fig. 16(a), the non-dimensional wavenumbers $k_2h$ and $k_3h$ corresponding to flexural and shear modes derived from both WSFEM ($N = 22$ and 6) and FSFEM are plotted for $\Delta t = 2$ μs. For the present beam configuration, the cut-off frequency for shear mode $k_3$ is given by $f_0 = \sqrt{GAK/\rho I} = 62$ kHz and this can be seen from Fig. 16(a). Thus the shear mode will propagate only for loading with frequency content greater than $f_0$. Similarly, Fig. 16(b) shows the spectrum relation for $\Delta t = 8$ μs and up to $f_{nyq} = 64$ μs derived from non-periodic WSFEM.

The dispersion relation for Timoshenko beam are presented in Fig. 17. The non-dimensional group speeds $C_g/C_0$ as described in the earlier sections, are derived from WSFEM with $N = 22$ and compared with FSFEM and a $\Delta t = 1$ μs is considered. One of the significant differences between the Euler–Bernoulli and Timoshenko beam is the presence of the second propagating mode. The speeds are plotted up to $f_N$ which is 300 kHz here and it can be seen from Fig. 17 that in this frequency range the WSFEM and FSFEM results matches exactly except deviations at frequencies very near to $f_N$.

![Fig. 17. Dispersion relation $C_g/C_0$, $C_0 = \sqrt{EA/\rho A}$ for Timoshenko beam.](image-url)
A sinusoidal loading modulated at 110 kHz (greater than the cut-off frequency) is used to capture the simultaneous existence of the coupled modes and the load spectrum is presented in Fig. 16(a). In Figs. 18(a) and (b), the response due to the above load are presented. The transverse

![Graph](image_url)

Fig. 18. Transverse velocity measured at D (Fig. 6), due to sinusoidal pulse modulated at 110 kHz applied at C with $\Delta t = 2\mu s$ and $L = 2.0 m$ (a) FSFEM and WSFEM, $N = 22$ and (b) WSFEM, $N = 6$. 
velocity is measured at $D$ (see Fig. 6) at $L = 2$ m from $C$ where the load is applied in the transverse direction. The simulations are done with $\Delta t = 2$ $\mu$s and as discussed for elementary rod and Euler–Bernoulli beam, for this $\Delta t$, $f_N = 150$ kHz for $N = 22$ and $f_N = 90$ kHz for $N = 6$, respectively. Thus, WSFEM with $N = 22$ will predict the exact velocity and this is validated with Fig. 18(a) which shows that the solution matches exactly with the corresponding FSFEM solution. However, the solution obtained from WSFEM with $N = 6$ plotted in Fig. 18(b) shows spurious dispersion as expected. Here, no propagating mode is clearly visible.

Next, numerical experiments are performed to study the response due to broadband impulse load. The details of the loading has been described earlier. The load is applied at the tip of a fixed-free beam of dimension similar to the Euler–Bernoulli beam and the tip transverse velocity is measured. Unlike for Euler–Bernoulli beam here $\Delta t$ is taken even higher and is equal to 8 $\mu$s. In Fig. 16(b), the superimposed spectrum of the impulse load shows that its frequency content equal to 44 kHz is much greater than $f_N = 23$ kHz for $N = 6$ while slightly for $N = 22$ for which $f_N = 38$ kHz. The time domain response plotted in Fig. 19 shows that WSFEM solution for $N = 22$ matches well with FSFEM solution while the solution for $N = 6$ shows deviations.

9. Conclusions

Spectral element method is shown to be an efficient alternative of FE analysis of wave propagation problems and decreases the computational cost substantially. The conventional
Fourier spectral finite element however possess certain problems particularly for time domain analysis of undamped finite length structures. The novelty of the spectral element developed in this paper is that it uses wavelet transform to reduce the PDEs to ODEs. Due to the localised nature of Daubechies wavelet basis functions, WSFEM proves effective for wave propagation analysis of undamped finite length structures where FSFEM does not work well due to associated wrap around problem. Though FSFEM has limitations for time domain analysis, it can be directly used for various frequency domain analysis. In this paper, a correspondence is established between FSFEM and WSFEM for determining the wave characteristics, that is the spectrum and dispersion relation for a given waveguide. It is found that the WSFEM predicts exact wave behaviour up to a threshold frequency which is a fraction of the Nyquist frequency. The threshold frequency can be increased or decreased by suitably tuning the sampling rate or the order of the basis functions. The sampling time rate required in WSFEM can be a priori determined depending on the excitation frequency and order of bases to avoid the introduction of spurious modes in the analysis. The above statement has been validated by obtaining the response due modulated sinusoidal pulse and broadband impulse load using WSFEM with different sampling rates and order of basis. All the above experiments are presented for elementary rod, Euler–Bernoulli and Timoshenko beam and comparison is presented with FSFEM.

References