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Parikshit Sonekar and Mira Mitra

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A wavelet-based model of one-dimensional periodic structure for wave-propagation analysis

By Parikshit Sonekar and Mira Mitra*

Department of Aerospace Engineering, Indian Institute of Technology Bombay, Mumbai 400076, India

In this paper, a wavelet-based method is developed for wave-propagation analysis of a generic multi-coupled one-dimensional periodic structure (PS). The formulation is based on the periodicity condition and uses the dynamic stiffness matrix of the periodic cell obtained from finite-element (FE) or other numerical methods. Here, unlike its conventional definition, the dynamic stiffness matrix is obtained in the wavelet domain through a Daubechies wavelet transform. The proposed numerical scheme enables both time- and frequency-domain analysis of PSs under arbitrary loading conditions. This is in contrast to the existing Fourier-transform-based analysis that is restricted to frequency-domain study. Here, the dispersion characteristics of PSs, especially the band-gap features, are studied. In addition, the method is implemented to simulate time-domain wave response under impulse loading conditions. The two examples considered are periodically simply supported beam and periodic frame structures. In all cases, the responses obtained using the present periodic formulation are compared with the response simulated using the FE model without the periodicity assumption, and they show an exact match. This validates the accuracy of the periodic assumption to obtain the time- and frequency-domain wave responses up to a high-frequency range. Apart from this, the proposed method drastically reduces the computational cost and can be implemented for homogenization of PSs.

Keywords: periodic structures; wave propagation; Daubechies wavelet; finite element

1. Introduction

Research on periodic structures (PSs), particularly the study of their dynamic properties, dates back to the eighteenth century (Brillouin 1953) and started mostly to understand wave characteristics of lattice structures. The interesting feature shown by PSs is the band-gap phenomenon, i.e. they exhibit certain frequency bands within which no wave propagates, resulting in filtering capabilities.

In the early Seventies and before, research activities were directed towards engineering PSs (Mead 1970; Sengupta 1970; Mallik & Mead 1977; Mead & Mallik 1978), such as periodically supported beams, rings and plates. Since then, a lot of work on modelling and analysis of PSs has been reported in the literature (Mead & Mallik 1976; Rao & Mallik 1977; Singh & Mallik 1977; Li & Benaroya 1992; Langley 1996; Gonella & Ruzzene 2008a,b; Mead 2009). This is primarily because

*Author for correspondence (mira@aero.iitb.ac.in).
such structures are often encountered in engineering, for example, a periodically stiffened fuselage and wing of an aircraft, a periodic layered media, a bridge with repeated truss-like structures, a periodically supported pipe conveying fluid and a PS under random loading. In addition, their band-gap characteristics are used in applications such as vibration attenuation (Asiri & Baz 2006; Ruzzene & Airoldi 2008) and piezoelectric transducers (Ballandras et al. 2003; Hofer et al. 2006). Recently, with the advent of nano-structured materials, for example, carbon nanotubes and their composites, the study of their periodic properties will help in exploring their functional properties.

This paper aims at developing a generalized numerical scheme based on Floquet theorem (Mead 1970) to simulate the overall wave response of one-dimensional multi-coupled linear PSs in both time and frequency domains. These simulations will help to understand the dynamic characteristics for different applications, as mentioned earlier. In addition, it will validate the periodicity assumption to simulate wave response in both time and frequency domains. The proposed scheme is based on transformation using the Daubechies scaling function (Daubechies 1992). These functions are bounded in both the time and frequency domains. The localized nature of the functions in time allows accurate simulation of time-domain responses, unlike the Fourier-transform-based methods that are restricted only to frequency-domain analysis. However, the localization in the time domain causes reduced accuracy in the frequency domain. As a result, in this wavelet-based method, the frequency-dependent wave characteristics can be obtained accurately only up to a certain fraction of the Nyquist frequency (Mitra & Gopalakrishnan 2006). It should be mentioned here that scaling functions corresponding to other wavelets with localized support and preferably orthogonal properties can be used for such modelling. The formulation, however, will be different from the present method in certain aspects.

The formulation starts with obtaining the dynamic stiffness matrix $K_D$ of a periodic cell shown in figures 1 and 2. Here, this matrix is obtained in the wavelet domain in contrast to the conventional dynamic stiffness matrix that is obtained in the frequency domain through Fourier transform. In this paper, two different methods of deriving $K_D$ are described. In the first technique, $K_D$ is calculated from the mass and stiffness matrices obtained from the finite-element (FE) method followed by wavelet transform. The second method is the wavelet-based spectral FE (WSFE) (Mitra & Gopalakrishnan 2005) technique, which directly formulates $K_D$ in the wavelet domain, starting from the governing partial differential equations. In brief, the WSFE method follows the FE procedure in the transformed wavelet domain. While the second method is computationally efficient and can model waveguides of rather high complexities, the FE method can model waveguides of any arbitrary geometry. The FE method for such problems, however, involves huge computational cost.
Next, the periodicity of the structure is considered using the Floquet theorem to develop the numerical scheme for wave-propagation analysis. The method is described in detail in the following sections. In addition to enabling simultaneous time- and frequency-domain analysis, the proposed scheme substantially reduces the computational cost associated with the simulation of time-domain wave response of PSs of rather high complexities. As mentioned earlier, it also validates the use of the periodicity assumption for the accurate simulation of wave response owing to higher frequency excitations. Finally, the method can be extended for homogenization of PSs.

The method developed is then implemented first to obtain the dispersion relation, i.e. frequency dependence of the propagation constants for different examples of PSs. These include periodically simply supported Euler–Bernoulli and Timoshenko beams, periodic frame structures. Next, the time-domain wave responses of these structures owing to broad-band impulse loading are simulated. These responses, obtained through the periodicity assumption, are compared with the response simulated using the FE method without considering periodicity. The response obtained using the proposed method is referred to as the periodic solution in the remainder of this paper. Numerical experiments are also performed to emphasize the ability of the proposed method in modelling PSs with higher complexities. The example considered is that of the periodically simply supported Euler–Bernoulli beam with localized defects.

This paper is organized as follows. In the next section, two different methods of obtaining the dynamic stiffness matrix of a periodic cell in the wavelet domain are explained. A brief introduction to the Daubechies example is also presented in this section. Section 3 explains, in detail, the mathematical formulation of the scheme considering the periodicity condition. In §§4 and 5, the numerical results are presented, as explained earlier. This paper ends with important conclusions and scope for future work.

2. Dynamic stiffness matrix

As mentioned earlier, the first step in the formulation of the proposed wavelet-based numerical scheme for PSs is to obtain the dynamic stiffness matrix $K_D$ of the periodic cell. The first method uses the FE method (Cook & Malkus 2007) to obtain the stiffness and mass matrices and the associated equation
of motion (EOM) as an ordinary differential equation (ODE) in time. Next, the Daubechies scaling function (Daubechies 1992) transformation is used in time to derive $K_D$ in the wavelet domain. The method is described briefly in §2b. FE modelling for wave-propagation analysis is, however, computationally costly and sometimes prohibitive. This is because the mesh size should be small enough to accurately capture the higher modes of vibration participating in wave propagation. The second method that can be implemented to obtain $K_D$ is the WSFE technique developed by the author (Mitra & Gopalakrishnan 2005, 2006). The WSFE method is tailored especially for wave-propagation analysis in finite-dimensional structural waveguides of higher complexities and is computationally very efficient. The method is explained in §2c. However, prior to these, a very brief introduction to Daubechies wavelets is provided in §2a.

(a) Daubechies compactly supported wavelets

A concise review of the orthogonal basis of Daubechies wavelets (Daubechies 1992) is provided. Wavelets $\psi_{j,k}(t)$ form compactly supported orthonormal bases for $L^2(\mathbb{R})$. The wavelets and the associated scaling functions $\varphi_{j,k}(t)$ are obtained by translation and dilation of single functions $\psi(t)$ and $\varphi(t)$, respectively,

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^jt - k), \quad j, k \in \mathbb{Z}$$

(2.1)

and

$$\varphi_{j,k}(t) = 2^{j/2}\varphi(2^jt - k), \quad j, k \in \mathbb{Z}.$$  

(2.2)

The scaling functions $\varphi(t)$ are derived from the dilation or scaling equation

$$\varphi(t) = \sum_k a_k \varphi(2t - k),$$

(2.3)

and the wavelet function $\psi(t)$ is obtained as

$$\psi(t) = \sum_k (-1)^k a_{1-k} \varphi(2t - k),$$

(2.4)

where $a_k$ are the filter coefficients and are fixed for a specific wavelet or scaling function basis. For compactly supported wavelets, only a finite number of $a_k$ are non-zero. The filter coefficients $a_k$ are derived by imposing certain constraints on the scaling functions that are as follows: (i) the area under the scaling function is normalized to 1, (ii) the scaling function $\varphi(t)$ and its translates are orthonormal, and (iii) the wavelet function $\psi(t)$ has $M$ vanishing moments. The number of vanishing moments $M$ denotes the order $N$ of the Daubechies wavelet, where $N = 2M$.

Let $P_j(f)(t)$ be the approximation of a function $f(t)$ in $L^2(\mathbb{R})$ using $\varphi_{j,k}(t)$ as the basis, at a certain level (resolution) $j$, then

$$P_j(f)(t) = \sum_k c_{j,k} \varphi_{j,k}(t), \quad k \in \mathbb{Z},$$

(2.5)

where $c_{j,k}$ are the approximation coefficients.
(b) Method 1: dynamic stiffness matrix from the finite-element method

FE modelling for structural dynamics problems generally provides an ODE involving the mass $M$ and stiffness matrices $K$, relating the displacement vector $u$ containing the nodal degrees of freedom (d.f.) that are arbitrarily referred to as $u$. Similarly, $F$ is the force vector containing nodal forces $F$. For an undamped structure, the ODE representing the EOM is obtained as

$$M \ddot{u} + Ku = F. \quad (2.6)$$

Here, $M$ and $K$ represent the mass and stiffness matrices of the periodic cell. Let each of the displacements $u(t)$ of the displacement vector $u$ (and similarly, force $F(t)$ of the force vector $F$) be considered discretized at $n$ points in the time window $[0, t_f]$. Let $\tau = 0, 1, 2, \ldots, n - 1$ be the sampling points, then

$$t = \Delta t \tau, \quad (2.7)$$

where $\Delta t$ is the time interval between two sampling points. The function $u(t)$ (similarly, $F(t)$) can be approximated or transformed by scaling function $\varphi(\tau)$ at an arbitrary scale as

$$u(t) = u_\tau(\tau) = \sum_k \hat{u}_k \varphi(\tau - k), \quad k \in \mathbb{Z}, \quad (2.8)$$

where $\hat{u}_k$ are the approximation coefficients. These approximation coefficients can be derived by multiplying both sides of equation (2.8) by the translates of the scaling function $\varphi(\tau - j)$. Next, using their orthogonality properties, we get

$$\hat{u}_j = \int u_\tau(\tau) \varphi(\tau - j) d\tau. \quad (2.9)$$

Similarly, considering acceleration $\ddot{u}(t)$ as an independent variable $v(t)$, it can be written as

$$v(t) = \ddot{u}(t) = \sum_k v_k \varphi(\tau - k) = \sum_k u_k \varphi''(\tau - k). \quad (2.10)$$

Again, multiplying both sides by the translates of $\varphi(\tau - k)$ and taking the inner product, we get

$$\hat{v}_j = \int v_\tau(\tau) \varphi(\tau - j) d\tau = \sum_{k=j-N+2}^{j+N-2} \hat{u}_k \Omega^2_{j-k}, \quad (2.11)$$

where $N$ is the order of the Daubechies scaling function as discussed earlier. $\Omega^2_{j-k}$ is the second-order connection coefficients defined as

$$\Omega^2_{j-k} = \int \varphi''(\tau - k) \varphi(\tau - j) d\tau. \quad (2.12)$$

For compactly supported wavelets, $\Omega^2_{j-k}(\tau - k)$ and also the other order of derivatives (Beylkin 1992) are non-zero only in the interval $k = j - N + 2$ to $k = j + N - 2$. While dealing with a finite-length data sequence, problems arise at the boundaries. It can be observed from equation (2.10) that certain coefficients $\hat{u}_j$ near the vicinity of the boundaries ($j = 0$ and $j = n - 1$) lie outside the time window defined by $j = 0, 1, 2, \ldots, n - 1$. Several approaches have been developed.
to handle these boundary conditions. Here, a wavelet extrapolation technique proposed by Amaratunga & Williams (1997) is implemented. The details of the implementation are also presented in a previous paper (Mitra & Gopalakrishnan 2005) by the author. This boundary treatment technique is found effective for time integration using wavelets.

After the treatment of the boundaries, equation (2.10) can be written as a matrix equation of the form

\[ \{ \hat{v}_j \} = \Gamma^2 \{ \hat{u}_j \}, \]  

where \( \Gamma^2 \) is the connection coefficient matrix obtained after imposing the boundary conditions. Through similar derivation, it can be shown that

\[ \Gamma^2 = \Gamma^{12}, \]  

where \( \Gamma^1 \) is the connection coefficient matrix corresponding to the first-order derivative. Next, the coupled matrix equation given by equation (2.13) can be decoupled through eigenvalue decomposition of the matrix \( \Gamma^2 \) as

\[ \Gamma^{12} = \Gamma^2 = \Phi \Pi \Phi^{-1}, \]  

where \( \Phi \) is the eigenvector matrix and \( \Pi \) is the diagonal eigenvalue matrix containing the eigenvalues that can be arbitrarily referred to as \(-\lambda_j^2\), with \(-i\lambda_j\) being the eigenvalues of \( \Gamma^1 \).

Substituting equation (2.15) into equation (2.13), we get

\[ \tilde{v}_j = \tilde{\tilde{u}}_j = -\lambda_j^2 \tilde{u}_j, \]  

where

\[ \tilde{u}_j = \Phi^{-1} \hat{u}_j \quad \text{and} \quad \tilde{v}_j = \Phi^{-1} \hat{v}_j. \]  

Now, multiplying both sides of equation (2.6) by the translates of the scaling functions and taking the inner product, we get

\[ M \ddot{\tilde{u}}_j + K \tilde{u}_j = \tilde{F}_j. \]  

Substituting equation (2.17) into equation (2.18), it can be written as

\[ -\lambda_j^2 M \tilde{u}_j + K \tilde{u}_j = \tilde{F}_j, \]  

or

\[ \begin{aligned} (K - \lambda_j^2 M) \tilde{u}_j &= \tilde{F}_j, & j = 0, 1, 2, \ldots, n - 1 \\ \end{aligned} \]

and

\[ \text{K}_{\text{Dj}} \tilde{u}_j = \tilde{F}_j. \]

Here, \( \lambda_j \) represents the circular frequency \( \omega_j \) used in conventional dynamic stiffness matrix. \( \lambda_j \) matches \( \omega_j \) exactly, but only up to a certain fraction \( p_N \) of the Nyquist frequency \( \omega_{\text{nyq}} \) (Mitra & Gopalakrishnan 2006).

(c) Method 2: dynamic stiffness matrix from the wavelet-based spectral finite-element method

In the previous subsection, the details of formulating \( \text{K}_{\text{Dj}} \) from the FE method in the wavelet domain have been described. As mentioned earlier, \( \text{K}_{\text{Dj}} \) can also be obtained using the WSFE method. In this subsection, the method
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(Mitra & Gopalakrishnan 2005, 2006) is described briefly and in a generalized way considering the example of an isotropic Timoshenko beam. Here, $K_{Dj}$ is obtained directly from the governing elasto-dynamic equations of the structure. The governing equations of a Timoshenko beam are given as

$$GA \left[ \frac{\partial^2 w}{\partial x^2} - \frac{\partial \phi}{\partial x} \right] = \rho A \frac{\partial^2 w}{\partial t^2}$$  \hspace{1cm} (2.21)

and

$$EI \frac{\partial^2 \phi}{\partial x^2} + GA \left[ \frac{\partial w}{\partial x} - \phi \right] = \rho I \frac{\partial^2 \phi}{\partial t^2},$$  \hspace{1cm} (2.22)

where $GA$ and $EI$ are the shear and bending stiffness, $\rho A$ and $\rho I$ are the corresponding inertias, $w(x, t)$ and $\phi(x, t)$ are the transverse displacements and rotations, respectively. The force boundary conditions are given as

$$GA \left[ \frac{\partial w}{\partial x} - \phi \right] = V$$ \hspace{1cm} (2.23)

and

$$EI \frac{\partial \phi}{\partial x} = M,$$ \hspace{1cm} (2.24)

where $V$ and $M$ are the shear force and bending moment, respectively.

Even here, first, the variables $w(x, t)$ and $\phi(x, t)$ are approximated using $\phi(t)$, similar to equation (2.8), as

$$w(x, t) = w_\tau(x, \tau) = \sum_k \hat{w}_k(x) \varphi(\tau - k)$$  \hspace{1cm} (2.25)

and

$$\phi(x, t) = \phi_\tau(x, \tau) = \sum_k \hat{\phi}_k(x) \varphi(\tau - k).$$

Here, however, the approximation coefficients $\hat{w}_k$ and $\hat{\phi}_k$ are a function of spatial location $x$. Next, equation (2.25) is substituted into equations (2.21) and (2.22). Following this, both sides of the equations are multiplied by the translates of $\varphi(t)$ and taking inner products on either sides gives

$$GA \left[ \frac{d^2 \hat{w}_j}{dx^2} - \frac{d \hat{\phi}_j}{dx} \right] = \rho A \sum_{k=j-N+2}^{j-N-2} \hat{w}_j \Omega_{j-k}^2$$  \hspace{1cm} (2.26)

and

$$EI \frac{d^2 \hat{\phi}_j}{dx^2} + GA \left[ \frac{d \hat{w}_j}{dx} - \hat{\phi}_j \right] = \rho I \sum_{k=j-N+2}^{j-N-2} \hat{\phi}_j \Omega_{j-k}^2.$$  \hspace{1cm} (2.27)
The above equations can be written in matrix form as
\[
GA \left[ \begin{array}{c}
\frac{d^2 \hat{w}}{dx^2} \\
\frac{d^2 \phi}{dx^2}
\end{array} \right] - \left[ \begin{array}{c}
\frac{d \hat{\phi}}{dx} \\
\frac{d \phi}{dx}
\end{array} \right] = \rho A \Gamma^{12} \{\hat{w}\} \tag{2.28}
\]
and
\[
EI \left\{ \frac{d^2 \hat{\phi}}{dx^2} \right\} + GA \left[ \begin{array}{c}
\frac{d \hat{w}}{dx} \\
\frac{d \phi}{dx}
\end{array} \right] - \{\hat{\phi}\} = \rho I \Gamma^{12} \{\hat{\phi}\}. \tag{2.29}
\]
These matrix equations can be decoupled through eigenvalue analysis as described earlier, and the decoupled form of equations (2.28) and (2.29) is of the form
\[
GA \left[ \begin{array}{c}
\frac{d^2 \tilde{w}_j}{dx^2} \\
\frac{d \tilde{\phi}_j}{dx}
\end{array} \right] = -\lambda^2_j \rho A \tilde{w}_j \tag{2.30}
\]
and
\[
EI \frac{d^2 \tilde{\phi}_j}{dx^2} + GA \left[ \begin{array}{c}
\frac{d \tilde{w}_j}{dx} \\
\frac{d \phi_j}{dx}
\end{array} \right] - \tilde{\phi}_j = -\lambda^2_j \rho I \tilde{\phi}_j. \tag{2.31}
\]
Similarly, the transformed and decoupled forms of the force boundary conditions given by equations (2.23) and (2.24) can be written as
\[
GA \left[ \begin{array}{c}
\frac{d \tilde{w}_j}{dx} \\
\frac{d \phi_j}{dx}
\end{array} \right] = \tilde{V}_j \tag{2.32}
\]
and
\[
EI \frac{d \tilde{\phi}_j}{dx} = \tilde{M}_j. \tag{2.33}
\]

The solutions of the decoupled ODEs given by equations (2.30) and (2.31) can be assumed as
\[
\tilde{w}_j = \sum_{l=1}^{4} C_j^l R_j^{1l} e^{-ik_j^l x} \tag{2.34}
\]
and
\[
\tilde{\phi}_j = \sum_{l=1}^{4} C_j^l R_j^{2l} e^{-ik_j^l x}. \tag{2.35}
\]
Substitution of the above assumed solution in equations (2.30) and (2.31) results in a polynomial eigenvalue problem (PEP), solution of which gives $R_j^{1l}$ and $R_j^{2l}$ as eigenvectors and $k_j^l$ as the eigenvalues. The constants $C_j^l$ are unknown and obtained from the displacement and force (equations (2.32) and (2.33)) boundary conditions at the two nodes of the WSFE method defined by $x = 0$ and $x = L$, with $L$ being the length of the element. Next, following the procedure of the FE method, the stiffness matrix relating the transformed nodal displacements to the transformed nodal forces given by equation (2.20) is obtained. This stiffness matrix is similar to the dynamic stiffness matrix $K_{Dj}$ in the wavelet domain.
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derived from the FE model, as described in the previous subsection. However, as the WSFE method is based on the exact solution of the governing differential equation, the transformed wavelet domain, one element is sufficient to model waveguides of arbitrary length in the absence of a discontinuity. This makes the simulations computationally efficient, unlike the FE solution, where a large number of elements are required to accurately predict the wave responses of a structure. For a structural assembly, the elemental $K_{Dj}$ obtained from the WSFE technique can be assembled to form the global dynamic stiffness matrices similar to the FE method.

3. Periodic formulation

In this section, the formulation to obtain the wave response of a PS using its periodic property is explained in detail. The formulation proceeds with the dynamic stiffness matrix $K_{Dj}$ for the periodic cell, as explained in the last section. $K_{Dj}$ relates the transformed nodal displacements $\tilde{u}_j$ and forces $\tilde{F}_j$. The subscript $j$ is dropped hereafter for simplified notation, and all the following equations should be evaluated for $j = 0, 1, 2, \ldots, n - 1$.

The elements of vector $\tilde{u}$ can be grouped into $\tilde{u}_L$, $\tilde{u}_R$ and $\tilde{u}_I$. Here, $\tilde{u}_L$, $\tilde{u}_R$ and $\tilde{u}_I$, respectively, represent the displacements of the l.h.s., r.h.s. and intermediate nodes of the PS. Similarly, the force vector $\tilde{F}$ is also clustered into $\tilde{F}_L$, $\tilde{F}_R$ and $\tilde{F}_I$. For a PS with intermediate nodes, the displacements corresponding to them ($\tilde{u}_I$) are condensed out such that $\tilde{u}$ contains only $\tilde{u}_L$ and $\tilde{u}_R$, which are of length $M$ each. Thus, for any arbitrary periodic cell, equation (2.20) can be written as

$$
\begin{bmatrix}
K_{LL}^{D} & K_{LR}^{D} \\
K_{RL}^{D} & K_{RR}^{D}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_L \\
\tilde{u}_R
\end{bmatrix} =
\begin{bmatrix}
\tilde{F}_L \\
\tilde{F}_R
\end{bmatrix}.
$$

(3.1)

For a PS, $\tilde{u}_L$ and $\tilde{u}_R$ are related as

$$
\tilde{u}_R = e^{-\xi d} \tilde{u}_L = e^{-\mu} \tilde{u}_L,
$$

(3.2)

where $d$ is the length of the periodic cell and $\mu$ is generally referred to as the propagation constant. Similarly, $\tilde{F}_L$ and $\tilde{F}_R$ follow the relation

$$
\tilde{F}_R = -e^{-\mu} \tilde{F}_L.
$$

(3.3)

Substituting equations (3.2) and (3.3) into equation (3.1), we can pose a PEP as

$$
\begin{bmatrix}
K_{RL}^{D} + (K_{LL}^{D} + K_{RR}^{D}) e^{-\mu} + K_{LR}^{D} e^{-2\mu}
\end{bmatrix} \tilde{u}_L = 0.
$$

(3.4)

The solution of the eigenvalue problem provides an eigenvector matrix $\Lambda$ and a diagonal matrix with the eigenvalues $e^{-\mu}$ as the diagonal terms. Since equation (3.4) is a quadratic equation, the eigenvalues will be a set of complex pairs corresponding to $\pm \mu_R \pm i\mu_I$.

In addition, owing to the periodic nature, equation (3.2), written for a single periodic cell, can be extended for a PS with $N_c$ number of cells as

$$
\tilde{u}_R = e^{-\xi N_c d} \tilde{u}_L = e^{-\mu N_c} \tilde{u}_L.
$$

(3.5)
For similar reasons, the generalized solution of the displacement $\mathbf{\tilde{u}}_{N_x}$ at the l.h.s. boundary of the $N_x$th cell of a PS can be written as

$$
\mathbf{\tilde{u}}^m_{N_x} = \sum_{l=1,3,5,\ldots}^{2M-1} C^l \Lambda^{ml} e^{-\mu^l N_x} + \sum_{l=2,4,6,\ldots}^{2M} C^l \Lambda^{ml} e^{-\mu^l (N_c - N_x)}, \quad (3.6)
$$

where $\mathbf{\tilde{u}}^m_{N_x}$ is the $m$th element of the vector $\mathbf{\tilde{u}}_{N_x}$ representing the $m$th d.f. and $C^l$ are the unknown constants to be derived from the boundary conditions. The particular form of the solution shown in equation (3.6) is considered to avoid ill-conditioning of the matrices. For a single cell, $N_c = 1$ in equation (3.6), with $N_x = 0$, giving $\mathbf{\tilde{u}}_L$ and $N_x = 1$, representing $\mathbf{\tilde{u}}_R$. Similarly, for a PS with the number of cells equal to $N_c$, substituting $N_x = 0$ into equation (3.6) gives the displacements $\mathbf{\tilde{u}}_L$ at the l.h.s. boundary and $N_x = N_c$ gives the displacements $\mathbf{\tilde{u}}_R$ at the r.h.s. boundary. Equation (3.6) can be written in matrix form as

$$
\mathbf{\tilde{u}}_{N_x} = \Lambda \Theta \{C\}, \quad (3.7)
$$

where $\Theta$ is a diagonal matrix with the diagonal terms $[e^{-\mu^1 N_x}, e^{-\mu^1 (N_c - N_x)}, e^{-\mu^2 (N_c - N_x)} \ldots e^{-\mu^1 N_x}, e^{-\mu^1 (N_c - N_x)} \ldots e^{-2M \mu^1 N_x}, e^{-2M \mu^1 (N_c - N_x)}]$. Substituting the boundary conditions at $N_x = 0$ and $N_x = N_c$ in the matrix equation given by equation (3.7), we get

$$
\mathbf{\tilde{u}}_L = \Lambda \Theta_{N_x=0} \{C\} = \mathbf{T}_{11} \{C\}, \quad (3.8)
$$

$$
\mathbf{\tilde{u}}_R = \Lambda \Theta_{N_x=N_c} \{C\} = \mathbf{T}_{12} \{C\} \quad (3.9)
$$

and

$$
\mathbf{\tilde{u}} = \begin{pmatrix} \mathbf{\tilde{u}}_L \\ \mathbf{\tilde{u}}_R \end{pmatrix} = \begin{bmatrix} \mathbf{T}_{11} \\ \mathbf{T}_{12} \end{bmatrix} \{C\} = [\mathbf{T}_1] \{C\}. \quad (3.10)
$$

Next, substituting equation (3.2) into the first row of equation (3.1), we can relate $\mathbf{\tilde{u}}_L$ to $\mathbf{\bar{F}}_L$ as

$$
\begin{bmatrix} \mathbf{K}_D^{LL} + e^{-\mu} \mathbf{K}_D^{LR} \end{bmatrix} \mathbf{\tilde{u}}_L = \mathbf{\bar{F}}_L. \quad (3.11)
$$

Substituting the generalized solution of the displacements given by equation (3.6) into the above equation (3.11), the force $\mathbf{\bar{F}}_{N_x}$ at the l.h.s. boundary of the $N_x$th cell of a PS can be written as

$$
\mathbf{\bar{F}}^m_{N_x} = \sum_{l=1,3,5,\ldots}^{2M-1} \sum_{p} (\mathbf{K}^{LL}_{D} + e^{-\mu^l} \mathbf{K}^{LR}_{D}) \Lambda^{pl} C^l e^{-\mu^l N_x}
$$

$$
+ \sum_{l=2,4,6,\ldots}^{2M} \sum_{p} (\mathbf{K}^{LL}_{D} + e^{-\mu^l} \mathbf{K}^{LR}_{D}) \Lambda^{pl} C^l e^{\mu^l (N_c - N_x)} \quad (3.12)
$$

The above equation can be written in a matrix form as

$$
\mathbf{\bar{F}}_{N_x} = \mathbf{\gamma} \Theta \{C\}, \quad (3.13)
$$
where \( \gamma^{ml} = \sum_p 2M (K_D^{LL}m_p + e^{-\mu l} K_D^{LR}m_p) A^p \). Thus, similar to the displacements, the force boundary conditions at \( N_x = 0 \) and \( N_x = N_c \) can be written as

\[
\tilde{F}_L = \gamma \Theta_{N_x=0} [C] = T_{21} [C], \tag{3.14}
\]

\[
\tilde{F}_R = \gamma \Theta_{N_x=N_c} [C] = T_{22} [C] \tag{3.15}
\]

and

\[
\tilde{F} = \begin{bmatrix} \tilde{F}_L \\ \tilde{F}_R \end{bmatrix} = \begin{bmatrix} T_{21} \\ T_{22} \end{bmatrix} [C] = [T_2] [C]. \tag{3.16}
\]

From equations (3.10) and (3.16), the transformed force \( \tilde{F} \) and displacement \( \tilde{u} \) can be related as

\[
\tilde{F} = T_2 T_1^{-1} \tilde{u} = T \tilde{u}. \tag{3.17}
\]

Knowing the force applied on the l.h.s. and r.h.s. boundaries of the PS, the displacements at these boundaries can be derived using the above equation. In addition, after the boundary displacements \( \tilde{u} \) are known, the constants \( \{ C \} \) can be obtained from equation (3.10). Substituting these \( \{ C \} \) in equation (3.7), one can obtain the displacement at any intermediate point. The displacement \( \tilde{u}_{N_c} \) in the time domain can be obtained through the inverse transforms of \( \tilde{u}_{N_c} \).

Thus, the proposed scheme reduces the problem of modelling PS with \( 2M \times (N_c + 1) \) d.f. to a problem with \( 2M \) d.f. This results in immense saving of computational costs. The computational efficiency of the present method is even more prominent for a PS with a large number of cells, e.g. carbon nanotubes and graphene sheets. The scheme can also be extended for homogenization of such structures. In addition, equation (3.17) shows a FE-like form that will enable modelling of the PS with higher complexities, for example, a PS with defects and a PS consisting of different periodic cells.

### 4. Frequency-domain analysis

In this section, the frequency-domain wave characteristics of the PS are studied. The parameter studied is the propagation constant, \( \mu \), for different periodic structural configurations. Two examples of such structures are considered. First is a periodically simply supported beam shown in figure 1. Second is a periodic frame structure shown in figure 2. The members of the frame are Euler–Bernoulli beams with three d.f., namely axial and transverse displacements, and rotation. In all the cases, the structural members are considered to be aluminium with \( E = 70 \text{ GPa}, \ v = 0.3 \) and \( \rho = 2700 \text{ kg m}^3 \). The cross-sectional area is \( A = 0.05 \times 0.01 \text{ m}^2 \).

In figure 3, the real (\( \mu_r \)) and imaginary (\( \mu_i \)) parts of \( \mu \) are plotted with respect to frequency up to 2.5 kHz for the simply supported beam. For such mono-coupled PSs, only one wave mode is present (Mead 1975), as seen in the figure. The length of each periodic cell, i.e. the length of the beam between two supports, is 0.5 m. The plots are obtained and compared for both Euler–Bernoulli and Timoshenko beam theories. The pattern of \( \mu \) is similar to that reported by Watanabe & Sugimoto (2005), for an articulated Euler–Bernoulli beam. As expected, \( \mu_r \) shows band gaps within which propagation will not occur. In the later part of the paper, the overall wave response of the structure owing to broadband impulse loads also...
Figure 3. (a) Real and (b) imaginary parts of the propagation constant ($\mu$) for a periodic simply supported beam with $d = 0.5$ m. (a) Black dots, Timoshenko beam; grey line, Euler-Bernoulli beam. (b) Black dots, Euler-Bernoulli beam; grey dots, Timoshenko beam.

exhibits this band-gap feature. $\mu_i$ also shows a similar trend; however, the non-zero band has a wider width compared to the real part. In addition, as $\mu$ has both real and imaginary parts, the waves are dispersive in nature, i.e. they will be attenuated as they propagate (Graff 1991). The comparison between the Euler-Bernoulli and Timoshenko beams shows a slight difference between the two, with the Timoshenko beam having a higher propagation speed. It can also be seen from figure 3a that the width of the band gap increases at higher frequencies. Numerical experiments were also performed for a similar beam, but with $d = 1.0$ m. For such a case, a higher number of band gaps than that for a beam with $d = 0.5$ m were observed within a given frequency range.

Figure 4 plots $\mu_r$ and $\mu_i$ for the frame shown in figure 2 with $d = 0.25$ m and the members as Euler-Bernoulli beams with three d.f., namely axial, transverse and rotational at each node. This represents a multi-coupled system with six d.f. coupling. The symmetry, however, results only in three unique wave modes, as observed from the figure. As mentioned earlier, $\mu$ is obtained numerically through the solution of a PEP, and hence it is difficult to separate out $\mu$ corresponding to each mode. The waves are, however, dispersive in nature owing to non-negligible imaginary components of $\mu$, and exhibit banded nature.
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Figure 4. (a) Real and (b) imaginary parts of the propagation constant ($\mu$) for a periodic frame structure of an Euler–Bernoulli beam with $d = 0.25\,\text{m}$.

5. Time-domain analysis

Here, the time-domain wave response of a PS owing to a broadband impulse load with different frequency contents is simulated using the proposed method. The responses obtained are also compared with the FE solution. The FE solution does not consider the periodicity condition. Hence, the FE model has a large system size, with the total number of d.f. $2M \times (N_c + 1)$. Similar to the frequency-domain analysis performed in the previous section, the examples of a periodic simply supported beam and truss with Euler–Bernoulli beam members are considered here. The loads are applied at one end point (A), and the responses are measured at the other end (B), shown in figures 1 and 2. Frequency content of the impulse load used is shown in figure 5. Similar impulse loads with a lower frequency content of 2.2 kHz have also been used for some of the examples presented.

In figure 6a, the time-domain responses of a periodic simply supported beam with $d = 0.25\,\text{m}$ owing to an impulse load of frequency content 4.4 kHz are plotted. The periodic simply supported beam is shown in figure 1 and has $N_c = 20$. The load is applied at A and the response is measured at point B. The response obtained considering periodicity is compared with the FE solution obtained.

Figure 5. Frequency content (4.4kHz) of the applied impulse load.

Figure 6. (a) Time history and (b) frequency response functions (FRFs) of the response of a periodic simply supported Euler–Bernoulli beam with $d = 0.25$ m and $N_c = 20$ for broadband impulse loads with frequency content 4.4kHz. (a,b) Black line, periodic; grey line, FE method.
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Figure 7. (a) Time history and (b) FRF of the transverse wave response of a periodic frame structure for a broadband load of 2.2 kHz. (a) Black line, periodic; grey line, FE method.

The periodic and FE solutions match exactly, and the corresponding frequency response functions (FRFs) are presented in figure 6b for a frequency range 0–3 kHz. The FRFs show three band gaps or stop bands between 0 and 3 kHz. This is in congruence with that predicted by the dispersion relation obtained from the frequency-domain analysis.

In conclusion, it can be said that, for periodically simply supported beam, the time-domain impulse response obtained through the consideration of periodicity matches the FE solution. The above-mentioned trends are observed for the periodic beam structure, irrespective of the number of cells, \( N_c \), and the length of each cell \( d \).

Next, the time-domain wave responses of the periodic truss structure with \( N_c = 10 \) and \( d = 0.125 \) m shown in figure 2 are studied. Each of the members of the truss is an Euler–Bernoulli beam with three d.f., as mentioned earlier. The impulse with a frequency content of 2.2 kHz is applied at A, and the wave responses are measured at B. The periodic responses are compared with the exact response.

In figure 7a, the transverse velocity, i.e. the velocity in the \( Y \)-direction is plotted. Even here, it can be seen that the periodic solution matches exactly with the exact solution. This is similar to the example of a simply supported periodic
beam. Other examples of a PS with or without kinematic constraints have been studied, and it has been observed that the periodic formulation gives the exact solution for structures. The corresponding FRFs are presented in figure 7\(b\) for a frequency between 0 and 3 kHz. Here, two stop bands are observed between approximately 0.9–1.4 and 2.15–2.4 kHz.

Finally, to illustrate the capability of the proposed method to model a PS of higher complexities, it is implemented to model a PS with a localized defect. The example considered is the periodically simply supported Euler–Bernoulli beam, as explained earlier and shown in figure 1. The length of each cell is \(d = 0.25\) m and the load applied is the impulse with a frequency content of 4.4 kHz. To simulate the defect, one cell is considered to have a reduced cross-sectional area as compared to the other cells. For this example, a 20 per cent reduction is considered. Using the present method, the problem had only \(2M \times 4\) d.f. as compared to \(2M \times (N_c + 1)\) d.f. associated with the total FE modelling without the periodicity assumption. Although here, the example of a PS with a local defect is adopted, it should be mentioned that the developed method can model a PS with other forms of complexities with much computational savings.
Figure 9. (a) Time history and (b) FRF of the response of a periodically simply supported Euler–Bernoulli beam with $N_c = 21$ and localized defects. (a,b) Black line, without defect; grey line, defect at centre; black dotted line, defect at end.

In figure 8a, the wave response measured at B owing to the load applied at A in the periodic beam with $N_c = 11$ is plotted. The response is compared with that of the beam with a defect cell at the centre. Comparison is also made with the response of the beam with the defect cell located two cells away from A. It can be observed from figure 8a that the difference in the response occurs owing to the presence of the defect. In figure 8b, the FRFs of the above responses are presented for the frequency range of 0–3 kHz. It can be seen that the differences are more prominent in the FRFs. The FRFs of the beam with defects show additional band gaps that are not present in the beam without defects. In the beams with defects, no natural frequency is observed in the range 2.05–2.5 kHz, unlike the beam without defects, which shows two natural frequencies in this frequency range.

Time histories and FRFs of the wave responses of a similar periodic beam, as explained in the last example, are plotted in figure 9a,b, respectively; however, here $N_c = 21$. The motivation of this example was to observe the effect of damage in a comparatively larger PS than in the last example. As expected, the difference
in the response caused by the defect is much smaller for a larger PS. The band-gap patterns observed from the FRFs shown in figure 9b are also not altered by the presence of the defect for a beam with \( N_c = 21 \), although the amplitude of the natural frequencies has been affected.

6. Conclusions

Here, a generalized Daubechies wavelet-based numerical tool is developed for wave-propagation analysis in one-dimensional multi-coupled PSs. The method described starts from the dynamic stiffness matrix of the periodic cell and can simulate time and frequency-domain responses of the structure using the periodicity condition. In this paper, the dynamic stiffness matrix is obtained in the wavelet domain using a Daubechies scaling function transform. Such matrices can be obtained using the FE method to derive the stiffness and mass matrices followed by a wavelet transform. The alternate method is to use the wavelet spectral FE technique developed by the author, which directly derives the required dynamic stiffness matrix. The necessity of using a wavelet transform in obtaining the dynamic stiffness matrix of the cell and also modelling of the PS is that it allows simultaneous frequency and time-domain analyses, unlike most of the other techniques available.

The proposed method is implemented to study the wave behaviour in periodically simply supported beam and periodic frame structures. In all the cases, the dispersion relation is obtained and these plots show the band-gap nature of PSs. The dispersion plot of a periodically simply supported beam is similar to that reported in the literature. Next, the time-domain responses owing to broadband impulse loads are simulated for the beam and the frame structures. The response of the beam and the frame structure obtained from the periodic and FE solutions of the entire PS without considering the periodicity is the same. This validates the suitability of using the periodicity condition defined by the Floquet theorem in accurately simulating time- and frequency-domain wave responses. Next, the method was implemented to study the wave response of PSs with localized defects and to compare it with the response of the structures without defects. Apart from analysing the effects of the defect on the wave response, the motivation of these examples was to illustrate the efficiency of the proposed scheme in handling PSs of higher complexities without losing computational efficiency.

The technique will have important applications, in the homogenization of PSs, which will be studied in future work. Furthermore, the method can be extended to model two-dimensional PSs.

References


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