SMALL-TIME LOCAL CONTROLLABILITY AND STABILIZABILITY OF SPACECRAFT ATTITUDE DYNAMICS UNDER CMG ACTUATION*

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Abstract. The purpose of this paper is to examine whether the presence of singular CMG configurations poses an obstruction to the small-time local controllability (STLC) and stabilizability of the attitude dynamics of a spacecraft actuated by single-gimbal CMGs. We identify a class of singular CMG configurations called critically singular configurations, at which the linearized dynamics fail to be controllable. Linear controllability, and hence STLC and stabilizability, hold at equilibria in which the CMG array is in a nonsingular or singular, but not critically singular, configuration. We give sufficient conditions and necessary conditions on critically singular configurations for STLC to hold. In particular, we show that STLC and stabilizability fail to hold at an equilibrium if the magnitude of the total angular momentum of the CMG array has a local maximum at the equilibrium.

Key words. Attitude control, Local controllability, Stabilizability, Control moment gyroscopes

AMS subject classifications. 93B05, 93D15, 93B29, 34H15

1. Introduction.

Control moment gyroscopes (CMGs) form an important class of torque actuators for spacecraft attitude control. Because of their large torque producing capability and smooth operation, CMGs are preferred over other torque actuators in applications involving precision pointing or momentum management of large, long duration spacecraft.

A single gimbal CMG comprises of an axisymmetric rotor spinning at a constant rate. The rotor axis is mounted on a gimbal. The orientation of the rotor axis can be changed by applying torque to the gimbal, and the resulting reaction torque serves to control the spacecraft attitude. The magnitude of torque produced by the CMG is a function of the constant rotor speed and gimbal rotation rate, while the direction is orthogonal to the rotor and gimbal axes. Clearly, a single CMG cannot produce torques in all directions. In order to obtain torques along three independent directions, as well as for actuation redundancy, CMGs are used in arrays consisting of multiple CMGs. An array of q single gimbal CMGs gives rise to a function $\nu : \mathbb{T}^q \to \mathbb{R}^3$ giving the components of the angular momentum of the CMG array as a function of the vector of q gimbal angles, which may be viewed as an element of the q-dimensional torus $\mathbb{T}^q$.

The actuation torque produced by the CMG array in a CMG configuration $\theta \in \mathbb{T}^q$ is given by [6]

$$-\nu'(\theta)\dot{\theta}, \tag{1.1}$$

where $\nu'(\theta)$ represents the differential of ν at θ, and the vector $\dot{\theta}$ of gimbal rates represents a tangent vector to $\mathbb{T}^q$ at θ. Equation (1.1) represents a kinematic map from the gimbal rates at the configuration $\theta \in \mathbb{T}^q$ to the actuation torque. A CMG

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array may thus be viewed as an input-output device whose inputs are gimbal rates and outputs are actuation torques given by (1.1). Accordingly, a common approach to attitude control using CMGs is to first compute the torque profiles needed to elicit the required spacecraft behavior, and then invert the kinematic map (1.1) to obtain the gimbal rates that produce those torque profiles.

Unfortunately, every CMG array possesses singular configurations at which the differential of ν loses rank and fails to be surjective. At such a singular configuration, there exists a *singular direction* such that no combination of gimbal rates yields an actuation torque along the singular direction, and the spacecraft dynamics become momentarily underactuated. In fact, given any direction, there exist singular configurations for which the given direction is a singular direction. A brief and accessible illustration of CMGs and singular configurations may be found in [6], while detailed expositions are given in [16], [26] and [15]. Since the common approach described in the paragraph above involves inverting the kinematic map (1.1), the approach encounters difficulties at singular configurations. Consequently, a considerable amount of research related to CMGs has focused on the geometry of singular configurations [16, 3, 26, 15] as well as on steering algorithms to maneuver CMG arrays to produce desired torque profiles while avoiding singular configurations [25, 24, 18, 12, 14, 17, 4]. The central issue of concern in this entire body of work is the collection of torque profiles that the CMG array is capable of producing.

Although not theoretically guaranteed to always work, steering algorithms combined with the approach described above have been highly successful in practice. Yet, certain fundamental questions remain unanswered. The main problem posed by the presence of singular configurations is that it makes it more difficult to devise algorithms for generating gimbal motions that result in any given actuation torque profile. While certain control objectives such as attitude trajectory tracking clearly require the ability to generate arbitrary torque profiles, other control objectives such as asymptotic stabilization or state-to-state steering do not. Hence, it is not evident that singular configurations affect the ability to perform control tasks such as asymptotic stabilization or state-to-state steering. Indeed, one may view the problem of singular configurations as essentially that of local underactuation. In the wider systems and controls literature, underactuation is the norm rather than the exception, as the degrees of actuation in a system of interest are often fewer than the mechanical degrees of freedom, and almost always fewer than the number of state variables. Yet, underactuation is never seen as an obstruction to stabilizability or controllability. These observations lead us to ask exactly which system-theoretic properties are adversely affected by the presence of singular CMG configurations.

The question stated above cannot be answered by only considering the CMG array in isolation, and instead requires the application of system-theoretic tools to the combined dynamics of the spacecraft and the CMG array. The central issue of concern in such an approach would be the spacecraft behaviors that the CMG array is capable of producing. The number of research papers that take this alternative approach is very small. In fact, even the properties of the linearization of the combined dynamics at an equilibrium do not appear to have been reported. Reference [12] applied feedback linearization to the combined dynamics of the spacecraft and the CMG array, and gave a steering law to implement the corresponding input transformation. More recently, [6] used nonlinear controllability results to show that the combined spacecraft-CMG system can be steered between any two states possessing the same total angular momentum despite the presence of singular CMG configurations. In other words, the presence of singular configurations does not act as an obstruction to global controllability. In this paper, we use the setting provided by [6] to investigate whether the presence of singular configurations poses an obstruction to the small-time local controllability (STLC) and stabilizability of the spacecraft attitude dynamics at rest equilibria, that is, states in which the spacecraft and CMG gimbals do not rotate.

While the importance of stabilizability is widely acknowledged, the property of STLC is of practical importance too. This is because a lack of STLC at an equilibrium x implies that, given a sufficiently small bounds on the steering time and gimbal rates, there exists a terminal state y arbitrarily close to x such that a trajectory starting from x cannot be steered to the terminal state y without violating at least one of the bounds. Thus, a lack of STLC could lead to a situation where an arbitrarily small terminal reorientation requires either a relatively large amount of time, or relatively large gimbal rates, or both. On the other hand, STLC is the reasonable requirement that terminal states that are sufficiently close to x may be reached from x in arbitrarily small times using bounded gimbal rates.

In Section 2, we identify a special class of singular configurations called *critically singular configurations* at which the magnitude of the total angular momentum of the CMG array has a critical value. Equivalently, critically singular configurations are those singular configurations in which the total angular momentum of the CMG array and the singular direction are linearly dependent. We give examples to emphasize that our notion of a critically singular configuration is different from the notion of an external singularity that is prevalent in the literature. In Section 3, we show that the linearized dynamics at an equilibrium are controllable if and only if the corresponding CMG configuration is not critically singular. It immediately follows that the attitude dynamics of the spacecraft are STLC as well as smoothly stabilizable at those equilibria in which the CMG array is either in a non-singular configuration or in a singular, but not critically singular, configuration.

In Section 4, we give sufficient conditions on critically singular CMG configurations for STLC to hold at equilibria involving those configurations. We use the sufficient condition for STLC given in [8] to show that STLC holds at every rest equilibrium if the total angular momentum of the spacecraft and the CMG array is zero. When the total angular momentum of the spacecraft and the CMG array is nonzero, STLC holds at those rest equilibria in which the CMG array is either in a critically singular configuration that is not an external singularity, or in a critically singular configuration such that the individual CMG angular momenta that are orthogonal to the total angular momentum of the CMG array span a two-dimensional space. Interestingly, stronger sufficient conditions for STLC such as [7, Thm. 1.1] or its more widely used interpretation given in [19] do not apply in this case. The problem of STLC under CMG actuation thus represents a problem of practical significance that also possesses sufficient complexity to necessitate the fullest use of available results for its resolution. To the best of our knowledge, our work represents the first instance where the results of [8] are used in a problem motivated by a practical application.

In Section 5, we use the necessary condition given in [22] to show that STLC fails to hold at an equilibrium if the corresponding CMG configuration is critically singular and an external singularity. As a corollary, we show that STLC fails to hold at an equilibrium if the magnitude of the total angular momentum of the CMG array has a local maximum at that equilibrium. It immediately follows that STLC fails to hold at all equilibria if the spacecraft carries only one CMG.

In Section 6, we show that equilibria in which the magnitude of the total angular momentum of the CMG array is either a local maximum or a nonzero local minimum cannot be stabilized using continuous time-invariant feedback. It immediately follows that no equilibrium is stabilizable using continuous time-invariant feedback if the spacecraft carries only one CMG.

In summary, CMG configurations that are not critically singular pose no obstruction to STLC and asymptotic stabilizability, regardless of whether they are external or internal singularities. Even among critically singular configurations, a large class of internal singularities pose no obstruction to STLC, while external singularities rule out STLC, and local extremizers of the total angular momentum magnitude rule out stabilizability.

We begin by introducing the necessary terminology, notation and description of the dynamics in Section 2.

2. Preliminaries.

2.1. Notation. Let SO(3) denote the Lie group of 3×3 special orthogonal matrices. The Lie algebra so(3) of SO(3) is the set of 3×3 real skew-symmetric matrices with the matrix commutator as the bracket operation. Define $(\cdot)^\times : \mathbb{R}^3 \to \mathrm{so}(3)$ by

$$a^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \ a \in \mathbb{R}^3.$$

For every $a \in \mathbb{R}^3$, $a^\times$ is simply the matrix representation in the standard basis of the linear map $b \mapsto (a \times b)$ on $\mathbb{R}^3$, where $\times$ denotes the usual cross product on $\mathbb{R}^3$. The map $(\cdot)^\times$ is linear, bijective, and satisfies $(a \times b)^\times = a^\times b^\times - b^\times a^\times$ for every $a, b \in \mathbb{R}^3$. The tangent space to SO(3) at $R \in \mathrm{SO}(3)$ is given by $\mathrm{T}_R\mathrm{SO}(3) = \{RG : G \in \mathrm{so}(3)\} = \{Rg^\times : g \in \mathbb{R}^3\}$.

We denote the Euclidean norm on $\mathbb{R}^3$ by $\|\cdot\|$ and the two-dimensional unit sphere $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ by S^2. We use S^1 to denote the unit circle. Given an integer $q > 0$, we let $\mathbb{T}^q = \mathrm{S}^1 \times \cdots \times \mathrm{S}^1$ denote the q-dimensional torus, the Cartesian product of q copies of S^1, and $\mathbb{I}_q$ denote the index set $\{1, \ldots, q\}$.

Given two C^∞ vector fields f and g on a C^∞ manifold $\mathcal{N}$, we denote their Lie bracket by $[f, g]$. If $\mathcal{N}$ is an embedded submanifold of a C^∞ manifold $\mathcal{M}$, and $\hat{f}$ and $\hat{g}$ are C^∞ extensions to $\mathcal{M}$ of the vector fields f and g, respectively, then $[f, g]$ is the restriction of $[\hat{f}, \hat{g}]$ to $\mathcal{N}$. In particular, if $\mathcal{M} = \mathbb{R}^n$ for some n, then, for every $x \in \mathcal{N} \subseteq \mathcal{M}$, the canonical identification between $\mathrm{T}_x\mathcal{M}$ and $\mathbb{R}^n$ yields

$$[f, g](x) = \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} [\hat{g}(x + hf(x)) - \hat{f}(x + hg(x))]. \tag{2.1}$$

As in [6, 5], we will find it convenient to apply the formula (2.1) to vector fields defined on $\mathrm{SO}(3) \times \mathbb{T}^q$ for some q, which can be viewed as an embedded submanifold of $\mathbb{R}^n$ for $n = 9 + q$.

2.2. Singular CMG Configurations. Consider an array of $q > 0$ single-gimbal CMGs, each having an angular momentum of a constant, nonzero magnitude. We do not require all CMGs to be identical. In particular, we do not assume that the angular momenta of all the CMGs have the same magnitude. The body components of the spin angular momentum of the ith CMG give rise to a function $\nu_i : \mathrm{S}^1 \to \mathbb{R}^3$. We will denote the gimbal angle of the ith CMG by $\theta_i \in [0, 2\pi)$. Since each value of the gimbal angle determines a unique orientation of the gimbal, it will be convenient to view θ_i as an element of S^1 itself.

For each $i \in \mathbb{I}_q$ we denote by $\nu_i' : \mathrm{S}^1 \to \mathbb{R}^3$ and $\nu_i'' : \mathrm{S}^1 \to \mathbb{R}^3$ the first and second derivatives, respectively, of ν_i with respect to θ_i. Since the spin angular momentum of each CMG is constrained to move along a circle, it is easy to show that $\nu_i'(\theta_i)$ is orthogonal to $\nu_i(\theta_i)$ while $\nu_i''(\theta_i) = -\nu_i(\theta_i)$ for every $i \in \mathbb{I}_q$ and every $\theta_i \in \mathrm{S}^1$.

The body components of the total spin angular momentum of the CMG array defines a function $\nu : \mathbb{T}^q \to \mathbb{R}^3$ of the vector of gimbal angles $\theta = (\theta_1, \cdots, \theta_q) \in \mathbb{T}^q$ given by $\nu(\theta) = \nu_1(\theta_1) + \cdots + \nu_q(\theta_q)$. The range $\nu(\mathbb{T}^q)$ of ν is called the *momentum volume* of the CMG array, while the topological boundary of $\nu(\mathbb{T}^q)$ is referred to as the *momentum envelope* of the CMG array.

A *singular configuration* of the CMG array corresponds to a $\theta \in \mathbb{T}^q$ such that the span of the vectors $\{\nu_1'(\theta_1), \ldots, \nu_q'(\theta_q)\}$ has dimension less than three. We denote the set of singular configurations by $\mathcal{S} \subset \mathbb{T}^q$. It is clear that nonsingular configurations exist only if $q \geq 3$.

Given $v \in \mathrm{S}^2$, we let $\mathcal{S}_v = \{\theta \in \mathcal{S} : v^{\mathrm{T}}\nu_i'(\theta) = 0, i \in \mathbb{I}_q\}$. It is clear that, for every $\theta \in \mathcal{S}$, there exists $v \in \mathrm{S}^2$ such that $v^{\mathrm{T}}\nu_i'(\theta) = 0$ for all $i \in \mathbb{I}_q$. Thus $\mathcal{S} = \cup_{\nu \in \mathrm{S}^2}\mathcal{S}_v$. If $\theta \in \mathcal{S}$ is such that $\theta \in \mathcal{S}_v$, then v is a *singular direction* corresponding to the singular configuration θ. Except in the case where $v \in \mathrm{S}^2$ coincides with at least one of the gimbal axes of the CMGs, $\mathcal{S}_v$ is finite, and this makes it possible to use S^2 to locally parametrize $\mathcal{S}$ near a singular configuration. Since we do not make use of this parametrization in this paper, we refer the interested reader to [16] for details.

The magnitude of the total angular momentum of the CMG array gives rise to a smooth function $\eta : \mathbb{T}^q \to \mathbb{R}$ defined by $\eta(\theta) = \|\nu(\theta)\|^2$. A *critically singular configuration* of the CMG array is a critical point of the smooth function η which further satifies $\theta \in \mathcal{S}$. It is easy to verify that $\theta \in \mathbb{T}^q$ represents a critically singular configuration if and only $\nu^{\mathrm{T}}(\theta)\nu_i'(\theta) = 0$ for all $i \in \mathbb{I}_q$, and there exists $v \in \mathrm{S}^2$ satisfying $v^{\mathrm{T}}\nu_i'(\theta_i) = 0$ for all $i \in \mathbb{I}_q$. Hence the set $\mathcal{C}$ of critically singular configurations is given by $\mathcal{C} = \{\theta \in \mathcal{S} : \nu^{\mathrm{T}}(\theta)\nu_i'(\theta) = 0, i \in \mathbb{I}_q\}$. It is also easy to verify that, if $\theta \in \mathbb{T}^q$ satisfies $\|\nu(\theta)\| \neq 0$, then $\theta \in \mathcal{C}$ if and only if $\theta \in \mathcal{S}_v$ for $v = \|\nu(\theta)\|^{-1}\nu(\theta) \in \mathrm{S}^2$. As a result, critical points of the function η that correspond to nonzero critical values are critically singular configurations. If $\theta \in \mathbb{T}^q$ satisfies $\|\nu(\theta)\| = 0$, then θ is a critical point of the map η, but $\theta \in \mathcal{C}$ if and only if $\theta \in \mathcal{S}$ holds additionally. Finally, it is evident that, in the case $q = 1$, every CMG configuration is critically singular. Hence singular configurations that are not critically singular exist only if $q \geq 2$.

The results that we will present later involve only the notions introduced above. However, in order to examine the how STLC relates to notions already used in the literature, we will briefly introduce two more ways of qualifying singular configurations.

Consider $v \in \mathrm{S}^2$. The component of the total angular momentum of the CMG array along v gives rise to a function $\beta_v : \mathbb{T}^q \to \mathbb{R}$ given by $\beta_v(\theta) = v^{\mathrm{T}}\nu(\theta)$. It is easy to see that $\theta \in \mathcal{S}_v$ if and only if θ is a critical point of the function β_v. Furthermore, if $\theta \in \mathcal{S}_v$ is such that

$$v^{\mathrm{T}}\nu_i(\theta_i) > 0,\ i \in \mathbb{I}_q, \tag{2.2}$$

then the standard second-order sufficient condition for optimality implies that θ is a strict local maximizer for the function β_v. In fact, it turns out that a $\theta \in \mathcal{S}_v$ satisfying (2.2) is a strict global maximizer of β_v. Borrowing terminology from the literature on CMGs, we will call a singular configuration $\theta \in \mathcal{S}_v$ satisfying (2.2) an *external singularity.* The motivation for doing so comes from the optimality property mentioned above, by which $\nu(\theta)$ lies on the boundary of the momentum volume $\nu(\mathbb{T}^q)$ if $\theta \in \mathcal{S}_v$ satisfies (2.2). By contrast, a singular configuration $\theta \in \mathcal{S}_v$ is called an *internal singularity* if (2.2) fails to hold.

A singular configuration $\theta \in \mathcal{S}$ is said to be *inescapable* if for no $\epsilon > 0$ there exists a differentiable curve $\gamma : [0, \epsilon] \to \mathbb{T}^q$ satisfying $\gamma(0) = \theta$, $\nu(\gamma(t)) = \nu(\theta)$ for all $t \in [0, \epsilon]$, and $\gamma(\epsilon) \notin \mathcal{S}$. Thus, a singular configuration θ is *escapable*, that is, not inescapable, if and only if it is possible to reach a nonsingular configuration by starting from θ and using *null gimbal motions*, that is, gimbal motions that do not disturb the total angular momentum of the CMG array. The distinction between escapable and inescapable singularities is important when dealing with CMG steering algorithms, since null motions can be incorporated into singularity avoidance steering laws. However, as our results will show, the distinction is not important for STLC. Before moving on, we mention that all external singularities are inescapable [16, 3, 26], although the converse is not true. A weaker sufficient condition for a singular configuration to be inescapable may be found in [3, 26].

The following example illustrates the definitions introduced above, and also shows that the notion of a critically singular configuration introduced by us and the notion of an external singularity given in the literature are independent.

EXAMPLE 2.1. A common CMG array used in practice is the so called pyramid array consisting of four single-gimbal CMGs arranged such that the gimbal axis of each CMG is orthogonal to one inclined face of a pyramid on a square base (see [26, 3, 14, 4, 5] for a figure). Assuming that the inclined faces of the pyramid make an angle $\alpha \in (0, \frac{\pi}{2})$ with its base, each gimbal angle is defined to be zero when the angular momentum of the corresponding CMG points to the apex of the pyramid, each gimbal rotation is taken to be positive in the right hand sense along the outward normal to the corresponding pyramid face, and each CMG possesses unit angular momentum, then, for a suitable choice of coordinate axes, the individual CMG angular momentum vectors and their derivatives are given by

$$\begin{aligned} \nu_1(\theta_1) &= \begin{bmatrix} -\cos\alpha\cos\theta_1, & -\sin\theta_1, & \sin\alpha\cos\theta_1 \end{bmatrix}^{\mathrm{T}}, \\ \nu_2(\theta_2) &= \begin{bmatrix} -\sin\theta_2, & \cos\alpha\cos\theta_2, & \sin\alpha\cos\theta_2 \end{bmatrix}^{\mathrm{T}}, \\ \nu_3(\theta_3) &= \begin{bmatrix} \cos\alpha\cos\theta_3, & \sin\theta_3, & \sin\alpha\cos\theta_3 \end{bmatrix}^{\mathrm{T}}, \\ \nu_4(\theta_4) &= \begin{bmatrix} \sin\theta_4, & -\cos\alpha\cos\theta_4, & \sin\alpha\cos\theta_4 \end{bmatrix}^{\mathrm{T}}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \nu_1'(\theta_1) &= \begin{bmatrix} \cos\alpha\sin\theta_1, & -\cos\theta_1, & -\sin\alpha\sin\theta_1 \end{bmatrix}^{\mathrm{T}}, \\ \nu_2'(\theta_2) &= \begin{bmatrix} -\cos\theta_2, & -\cos\alpha\sin\theta_2, & -\sin\alpha\sin\theta_2 \end{bmatrix}^{\mathrm{T}}, \\ \nu_3'(\theta_3) &= \begin{bmatrix} -\cos\alpha\sin\theta_3, & \cos\theta_3, & -\sin\alpha\sin\theta_3 \end{bmatrix}^{\mathrm{T}}, \\ \nu_4'(\theta_4) &= \begin{bmatrix} \cos\theta_4, & \cos\alpha\sin\theta_4, & -\sin\alpha\sin\theta_4 \end{bmatrix}^{\mathrm{T}}. \end{aligned} \tag{2.4}$$

Consider the configurations

$$\theta^{\mathrm{A}} \stackrel{\mathrm{def}}{=} \begin{bmatrix} \pi/2 \\ \pi/2 \\ \pi/2 \\ \pi/2 \end{bmatrix}, \theta^{\mathrm{B}} \stackrel{\mathrm{def}}{=} \begin{bmatrix} 0 \\ 0 \\ \pi \\ \pi \end{bmatrix}, \theta^{\mathrm{C}} \stackrel{\mathrm{def}}{=} \begin{bmatrix} 0 \\ \pi \\ 0 \\ \pi \end{bmatrix}, \theta^{\mathrm{D}} \stackrel{\mathrm{def}}{=} \begin{bmatrix} -\pi/2 \\ \pi \\ \pi/2 \\ 0 \end{bmatrix}, \theta^{\mathrm{E}} \stackrel{\mathrm{def}}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.5}$$

The following assertions may be easily verified by using the expressions (2.3) and (2.4).

1. $\dim \operatorname{span}\{\nu_1'(\theta_1^{\mathrm{A}}), \ldots, \nu_4'(\theta_4^{\mathrm{A}})\} = 3$. Thus, $\theta^A \notin \mathcal{S}$. Also, $\nu(\theta^{\mathrm{A}}) = 0$, so that θ^{A} is a critical point of the function η. However, $\theta^{\mathrm{A}} \notin \mathcal{C}$ since $\theta^{\mathrm{A}} \notin \mathcal{S}$.
2. $v^{\mathrm{T}}\nu_i'(\theta_i^{\mathrm{B}}) = 0$ for $v = [0,0,1]^{\mathrm{T}}$ and $i \in \mathbb{I}_4$. Thus $\theta^{\mathrm{B}} \in \mathcal{S}$. However, $\nu(\theta^{\mathrm{B}})^{\mathrm{T}}\nu_i'(\theta_i^{\mathrm{B}})$ is nonzero for all $i \in \mathbb{I}_4$. Hence $\theta^{\mathrm{B}} \notin \mathcal{C}$. Moreover, $v^{\mathrm{T}}\nu_i(\theta_4^{\mathrm{B}}) > 0$ for all $i \in \mathbb{I}_4$, so that θ^{B} is an external singularity and an inescapable singularity. This shows that not all external or inescapable singularities are critically singular configurations.
3. $v^{\mathrm{T}}\nu_i'(\theta_i^{\mathrm{C}}) = 0$ for $v = [0,0,1]^{\mathrm{T}}$ and $i \in \mathbb{I}_4$. Thus $\theta^{\mathrm{B}} \in \mathcal{S}$. In addition, $\nu(\theta^{\mathrm{C}}) = 0$, so that $\theta^{\mathrm{C}} \in \mathcal{C}$.
4. $\nu(\theta^{\mathrm{D}})^{\mathrm{T}}\nu_i'(\theta_i^{\mathrm{D}}) = 0$ for all $i \in \mathbb{I}_4$, $\nu(\theta^{\mathrm{D}})^{\mathrm{T}}\nu_i(\theta_i^{\mathrm{D}}) > 0$ for all $i \in \{1,3\}$ and $\nu(\theta^{\mathrm{D}})^{\mathrm{T}}\nu_i(\theta_i^{\mathrm{D}}) < 0$ for all $i \in \{2,4\}$, so that θ^{D} is an internal singularity contained in $\mathcal{C}$. This also shows that not all critically singular configurations are external singularities.
5. $\nu(\theta^{\mathrm{E}})^{\mathrm{T}}\nu_i'(\theta_i^{\mathrm{E}}) = 0$ for all $i \in \mathbb{I}_4$, so that $\theta^{\mathrm{E}} \in \mathcal{C}$. In addition, $\nu(\theta^{\mathrm{D}})^{\mathrm{T}}\nu_i(\theta_i^{\mathrm{D}}) > 0$ for all $i \in \mathbb{I}_4$, so that θ^{E} is an external singularity contained in $\mathcal{C}$. Moreover, if $\sin^2\alpha > \frac{1}{2}$, then the Hessian matrix of the function η at θ^{E} is negative-definite, and hence θ^{E} is a local maximizer of η.

2.3. Attitude Dynamics. Consider a spacecraft carrying an array of $q > 0$ single gimbal CMGs in a torque-free environment. We describe the attitude of the spacecraft using a matrix $R \in \mathrm{SO}(3)$ such that the multiplication of the body components of a vector by R gives the components of that vector with respect to a reference inertial frame. Since no external torques act on the spacecraft, total inertial angular momentum of the spacecraft-CMG system is a constant. The dynamics of the spacecraft on a level set of the total inertial angular momentum may be written as

$$\dot{R}(t) = R(t)(J^{-1}\{R^{\mathrm{T}}(t)\mu - \nu(\theta(t))\})^{\times}, \tag{2.6}$$

where $J \in \mathbb{R}^{3\times 3}$ is the symmetric, positive-definite, moment-of-inertia matrix of the spacecraft about the body-fixed frame, and $\mu \in \mathbb{R}^3$ gives the constant inertial components of the total angular momentum of the spacecraft and the CMG array. See [6] for a detailed derivation of (2.6). Denoting the gimbal rate of the ith CMG by u_i, we may write

$$\dot{\theta}_i(t) = u_i(t),\ i \in \mathbb{I}_q. \tag{2.7}$$

Equations (2.6) and (2.7) describe the combined dynamics of the spacecraft and the CMG array when the total inertial angular momentum takes the constant value $\mu \in \mathbb{R}^3$, and define a μ-parametrized family of control systems of the form

$$\dot{y}(t) = f_\mu(y(t)) + g_1(y(t))u_1(t) + \cdots + g_q(y(t))u_q(t), \tag{2.8}$$

on the $3+q$-dimensional manifold $\mathcal{N} \stackrel{\mathrm{def}}{=} \mathrm{SO}(3) \times \mathbb{T}^q$, where $y = (R,\theta) \in \mathcal{N}$ represents the spacecraft attitude and the CMG configuration. The drift vector field f_μ and the control vector fields $g_1, g_2, \ldots, g_q$, are real analytic on $\mathcal{N}$, and are given by

$$f_\mu(R,\ \theta) = \left(R\left(J^{-1}\left\{R^{\mathrm{T}}\mu - \nu(\theta)\right\}\right)^{\times}, 0\right), \tag{2.9}$$

$$g_i(R,\ \theta) = (0, e_i), \tag{2.10}$$

where, for each $i \in \mathbb{I}_q$, $e_i \in \mathbb{R}^q$ is the ith column vector of the $q \times q$ identity matrix. We will assume that the gimbal rates are measurable functions of time constrained to take values in the compact *polydisk*

$$\mathcal{H}_\rho \stackrel{\text{def}}{=} \{u \in \mathbb{R}^q : |u_i| \leq \rho_i, i \in \mathbb{I}_q\}, \tag{2.11}$$

for some constants $\rho_i \in (0, \infty)$, $i \in \mathbb{I}_q$.

We denote the set of uncontrolled equilibria of (2.8) by $\mathcal{E}_\mu$. It is easy to see from (2.9) that $\mathcal{E}_\mu = \{(R, \theta) \in \mathcal{N} : R^{\mathrm{T}}\mu = \nu(\theta)\}$.

The dynamics (2.8)-(2.10) are said to possess the STLC property at an equilibrium $p \in \mathcal{E}_\mu$ if, for every $T > 0$, p lies in the interior of the set of states that can be reached in time not greater than T by following trajectories of (2.8) starting at p that result from using gimbal rates satisfying the constraint (2.11) [23].

Before moving on, we will briefly compare our formulation of the dynamics with previous formulations appearing in the literature. The recent paper [2] considers STLC of the dynamics described by (2.6) for a specific choice of a CMG array when the CMG gimbal angles are themselves treated as control inputs. However, standard controllability results assume a very general class of input functions and allow even discontinuities. In addition to being physically unrealistic for gimbal angles, such behavior would also cause the assumptions used to derive (2.6) to cease to hold. Our formulation is taken from [6], and our choice of gimbal rates as inputs is consistent with a major portion of the CMG literature. The idea of eliminating the angular velocity and formulating the dynamics on a level set of the inertial angular momentum first appears in [12], although the formulation in [12] used quaternion variables instead of rotation matrices. The formulation of [9, 21, 20] only includes the total CMG momentum among the state variables without accounting for the inner structure of the CMG array, and is therefore not suitable for investigating the effect of singular configurations on control-theoretic properties of the combined dynamics. Finally, we note that our formulation focuses only on the dynamics on an angular momentum level set, and is therefore not suited for studying disturbance rejection problems involving external torques.

3. Controllability of the Linearization. In this section, we analyze the controllability of the linearization of (2.8) at its equilibrium points. Our main result of this section depends on two lemmas, which we state next and prove in the appendix. The first of these concerns CMG configurations that are singular, but not critically singular.

LEMMA 3.1. *Suppose $\theta \in \mathbb{T}^q$ and $v \in \mathrm{S}^2$ are such that $\theta \in \mathcal{S}_v \backslash \mathcal{C}$. Then, there exist $i, j \in \mathbb{I}_q$ such that $v^{\mathrm{T}}(J^{-1}\nu_i'(\theta_i) \times \nu(\theta)) \neq 0$ and $\nu_i'(\theta_i) \times \nu_j'(\theta_j) \neq 0$.*

Before stating our next lemma, we list the expressions for a few brackets involving the drift and control vector fields of (2.8). These brackets were computed by locally treating $\mathcal{N}$ as an embedded submanifold of $\mathbb{R}^{3\times 3} \times \mathbb{R}^q$ and using (2.1). Recall that given a C^∞ vector field g on $\mathcal{N}$, the Lie bracket $\mathrm{ad}_{f_\mu}^n g$ is defined to be g for $n = 0$ and recursively defined by $\mathrm{ad}_{f_\mu}^n g = [f_\mu, \mathrm{ad}_{f_\mu}^{n-1} g]$ for $n > 0$. Given $\mu \in \mathbb{R}^3$, $x = (R, \theta) \in \mathcal{N}$ and $i, j \in \mathbb{I}_q$ such that $i \neq j$, we have

$$[g_i, g_j](x) = 0, \ [f_\mu, g_i](x) = (R(J^{-1}\nu_i'(\theta_i))^\times, 0), \tag{3.1}$$

$$[g_j, [f_\mu, g_i]](x) = 0, \ [g_i, [f_\mu, g_i]](x) = (-R(J^{-1}\nu_i(\theta_i))^\times, 0), \tag{3.2}$$

$$[[f_\mu, g_i], [f_\mu, g_j]](x) = (R(J^{-1}\nu_i'(\theta_i) \times J^{-1}\nu_j'(\theta_j))^\times, 0), \tag{3.3}$$

$$[[g_i, [f_\mu, g_i]], [f_\mu, g_i]](x) = (R(J^{-1}\nu_i'(\theta_i) \times J^{-1}\nu_i(\theta_i))^\times, 0). \tag{3.4}$$

$$\mathrm{ad}^2_{f_\mu} g_i(x) = (R\{J^{-1}(J^{-1}\nu_i'(\theta_i) \times R^{\mathrm{T}}\mu) + J^{-1}(R^{\mathrm{T}}\mu - \nu(\theta)) \times J^{-1}\nu_i'(\theta_i)\}^\times, 0), \tag{3.5}$$

$$[g_i, \mathrm{ad}^2_{f_\mu} g_i](x) = (-R\{J^{-1}(J^{-1}\nu_i(\theta_i) \times R^{\mathrm{T}}\mu) + J^{-1}(R^{\mathrm{T}}\mu - \nu(\theta)) \times J^{-1}\nu_i(\theta_i)\}^\times, 0), \tag{3.6}$$

If, in addition, $\mu = 0$, then

$$[\mathrm{ad}^1_{f_\mu} g_i, \mathrm{ad}^2_{f_\mu} g_i](x) = (R\{J^{-1}\nu_i'(\theta_i) \times (J^{-1}\nu_i'(\theta_i) \times J^{-1}\nu(\theta))\}^\times, 0). \tag{3.7}$$

Lemma 3.2. *Let $\mu \in \mathbb{R}^3$, and suppose $p = (R, \theta) \in \mathcal{E}_\mu$ is such that $\theta \in \mathcal{C}$. Then* $\dim \mathrm{span}\{\mathrm{ad}^n_{f_\mu} g_i(p) : i \in \mathbb{I}_q, n \geq 1\} \leq 2$. *In particular, if $\mu \neq 0$, then* $\mathrm{span}\{\mathrm{ad}^n_{f_\mu} g_i(p) : i \in \mathbb{I}_q, n \geq 1\} \subseteq \mathrm{span}\{(R(J^{-1}w)^\times, 0) : w \in \mathbb{R}^3, w^{\mathrm{T}}\nu(\theta) = 0\}$.

Our first result, which we state next, asserts that the linearization of (2.8) at an equilibrium is controllable if and only if the corresponding CMG configuration is not critically singular.

Theorem 3.3. *Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in \mathcal{E}_\mu$. Then the linearization of (2.8)-(2.10) at p is controllable if and only if $\theta \notin \mathcal{C}$.*

Proof. Note that the linearization of (2.8) at the equilibrium p is controllable if and only if the tangent vectors $\{\mathrm{ad}^n_{f_\mu} g_i(p) : n = 0, \ldots, q+2, i \in \mathbb{I}_q\}$ span $\mathrm{T}_p\mathcal{N}$ (see [13]).

To prove sufficiency, first suppose $\theta \notin \mathcal{S}$. Then there exist distinct $i, j, k \in \mathbb{I}_q$ such that $\nu_i'(\theta_i)$, $\nu_j'(\theta_j)$ and $\nu_k'(\theta_k)$ are linearly independent. It follows from (3.1) that $[f_\mu, g_i](p)$, $[f_\mu, g_j](p)$, and $[f_\mu, g_k](p)$ are linearly independent. These three tangent vectors together with the control vector fields $g_1(p), \ldots, g_q(p)$ span $\mathrm{T}_p\mathcal{N}$. As a result, the linearization of (2.8) at p is controllable.

Next, suppose $\theta \in \mathcal{S}\backslash\mathcal{C}$. Then there exists $v \in \mathrm{S}^2$ such that $v^{\mathrm{T}}\nu_i'(\theta_i) = 0$ for every $i \in \mathbb{I}_q$. Since $\theta \notin \mathcal{C}$, we may assume that v satisfies $v \times \nu(\theta) \neq 0$. It follows from Lemma 3.1 that there exist distinct $i, j \in \mathbb{I}_q$ such that $\nu_i'(\theta_i) \times \nu_j'(\theta_j) \neq 0$, and $v^{\mathrm{T}}[J^{-1}\nu_i'(\theta_i) \times \nu(\theta)] \neq 0$. Because $v^{\mathrm{T}}\nu_i'(\theta_i) = v^{\mathrm{T}}\nu_j'(\theta_j) = 0$, it follows that $\nu_i'(\theta_i)$, $\nu_j'(\theta_j)$ and $J^{-1}\nu_i'(\theta_i) \times \nu(\theta) = J^{-1}\nu_i'(\theta_i) \times R^{\mathrm{T}}\mu$ are linearly independent. Equations (3.1) and (3.5) imply that the tangent vectors $[f_\mu, g_i](p)$, $[f_\mu, g_j](p)$ and $[f_\mu, [f_\mu, g_i]](p)$ are linearly independent. These three tangent vectors when taken together with the control vector fields $g_1(p), \ldots, g_q(p)$ span $\mathrm{T}_p\mathcal{N}$. As a result, the linearization of (2.8) at p is controllable.

To prove necessity, suppose $\theta \in \mathcal{C}$. It follows from Lemma 3.2 that

$$\begin{aligned} &\dim \mathrm{span}\{\mathrm{ad}^n_{f_\mu} g_i(p) : 0 \leq n \leq q+2, i \in \mathbb{I}_q\} \\ &= q + \dim \mathrm{span}\{\mathrm{ad}^n_{f_\mu} g_i(p) : 1 \leq n \leq q+2, i \in \mathbb{I}_q\} \leq q+2. \end{aligned}$$

Consequently, the linearization of (2.8) at p is not controllable. □

Since controllability of the linearization at an equilibrium implies STLC at that equilibrium (see [11, Cor. 7.28]), it follows as a corollary of Theorem 3.3 that the dynamics (2.8) are STLC at an equilibrium in which the CMG array is not in a critically singular configuration. We state this corollary without proof.

COROLLARY 3.4. *Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in \mathcal{E}_\mu$ is such that $\theta \notin \mathcal{C}$. Then the dynamics described by (2.8)-(2.10) are STLC at p under the input constraint (2.11).*

EXAMPLE 3.5. Consider a spacecraft carrying the CMG array described in Example 2.1. Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in \mathcal{E}_\mu$. Theorem 3.3 and Corollary 3.4 imply that, if θ equals one of the configurations θ^{A} or θ^{B} introduced in Example 2.1, then the dynamics described by (2.8)-(2.10) possess a controllable linearization at p, and are STLC at p under the input constraint (2.11). On the other hand, the linearized dynamics at p are not controllable if θ equals any of θ^{C}, θ^{D} or θ^{E}. The results of subsequent sections will allow us to examine STLC at equilibria involving these three configurations.

4. Sufficient Conditions for STLC at Critically Singular Configurations. In this section, we consider the STLC property at equilibria in which the CMG array is in a critically singular configuration. The next theorem is our main result.

THEOREM 4.1. *Let $\mu \in \mathbb{R}^3$, and suppose $p = (R, \theta) \in \mathcal{E}_\mu$ is such that $\theta \in \mathcal{C}$. If either one of the following three conditions hold, then the dynamics described by (2.8)-(2.10) are STLC at p under the input constraint (2.11).*

$$\nu(\theta) = 0. \tag{4.1}$$

$$\min\{\nu_i(\theta_i)^{\mathrm{T}}\nu(\theta) : i \in \mathbb{I}_q\} < 0. \tag{4.2}$$

$$\left.\begin{array}{c}\min\{\nu_i(\theta_i)^{\mathrm{T}}\nu(\theta) : i \in \mathbb{I}_q\} = 0, \\ \dim \operatorname{span}\{\nu_i'(\theta_i) : i \in \mathbb{I}_q, \nu_i(\theta_i)^{\mathrm{T}}\nu(\theta) = 0\} = 2.\end{array}\right\} \tag{4.3}$$

Theorem 4.1 implies that, if either of (4.1)-(4.3) hold, then, given an arbitrarily small bound on the steering time, it is possible to steer the trajectory of (2.8)-(2.10) starting at p to all terminal states in a sufficiently small neighborhood of p within the stipulated time using gimbal motions satisfying the rate constraints (2.11). Before proceeding, it is worthwhile to consider the physical interpretations of the conditions (4.1)-(4.3). Since $p = (R, \theta) \in \mathcal{E}_\mu$, condition (4.1) is equivalent to the condition that the total angular momentum μ of the spacecraft and the CMG array is zero. Note that, if $\nu(\theta) \neq 0$, then $0 < \nu(\theta)^{\mathrm{T}}\nu(\theta) = \sum_{i=1}^{q} \nu(\theta)^{\mathrm{T}}\nu_i(\theta_i)$, so that $\max\{\nu_i(\theta_i)^{\mathrm{T}}\nu(\theta) : i \in \mathbb{I}_q\} > 0$, while $\nu(\theta) \neq 0$ is equivalent to $\mu \neq 0$. Hence (4.2) is equivalent to the condition that the total angular momentum μ is nonzero and the critically singular configuration θ is an internal singularity. It is sufficient to consider (4.3) only in the case $\nu(\theta) \neq 0$. In this case, $\nu_i(\theta_i)^{\mathrm{T}}\nu(\theta) = 0$ for some $i \in \mathbb{I}_q$ along with $\theta \in \mathcal{C}$ implies that $\nu(\theta)$ is colinear with the gimbal axis of the ith CMG. Hence (4.3) is equivalent to the condition that the torques generated by all those CMGs whose gimbal axes are colinear with the total CMG angular momentum vector $\nu(\theta)$ should span a two-dimensional subspace of $\mathbb{R}^3$. Note that Theorem 4.1 does not apply if $\theta \in \mathcal{C}$ is also an external singularity, since each of (4.1)-(4.3) is violated in that case.

We will use Proposition 5.1 of [8] to prove Theorem 4.1. A slightly more general result is Theorem 5.4 of [8], which is specialized to stationary trajectories in Theorem 7.20 of [11]. The application of these results will require us to introduce some additional terminology and notation. The reader may also refer to section 3 of [8], subsection 7.1.5 of [11], or section 1 of [7].

Let $\mathrm{Lie}(\xi)$ be the free Lie algebra generated by the noncommutative indeterminates $\xi = \{\xi_0, \xi_1, \ldots, \xi_q\}$. Elements of $\mathrm{Lie}(\xi)$ are linear combinations over $\mathbb{R}$ of formal Lie brackets of the indeterminates ξ. Let $\mathrm{Lie}_0(\xi)$ denote the smallest Lie subalgebra of $\mathrm{Lie}(\xi)$ containing $\{\xi_1, \ldots, \xi_n\}$ that is closed under Lie brackets with ξ_0. Given a bracket $B \in \mathrm{Lie}(\xi)$ and $i \in \{0\} \cup \mathbb{I}_q$, we denote by $|B|_i$ the number of times the indeterminate ξ_i appears in the bracket B. A bracket $B \in \mathrm{Lie}_0(\xi)$ is *bad* if $|B|_0$ is odd and $|B|_i$ is even for every $i \in \mathbb{I}_q$. Let $\mathcal{B}$ denote the subspace generated by all the bad brackets in $\mathrm{Lie}_0(\xi)$.

An *admissible weight vector* is a nonnegative vector $l = [l_0, l_1, \ldots, l_q]^{\mathrm{T}} \in \mathbb{R}^{q+1}$ such that $[\epsilon^{l_1 - l_0} x_1, \ldots, \epsilon^{l_q - l_0} x_q]^{\mathrm{T}} \in \mathcal{H}_\rho$ for every $\epsilon \in (0,1)$ and $x \in \mathcal{H}_\rho$. It is clear that, under our assumptions on the vector ρ, a nonnegative vector l is an admissible weight vector if and only if $l_i \geq l_0$ for all $i \in \mathbb{I}_q$.

Next, suppose l is an admissible weight vector. The l-degree of a bracket $B \in \mathrm{Lie}(\xi)$ is the sum $\sum_{i=0}^{q} l_i |B|_i$. The definition of the l-degree extends naturally to *l-homogeneous* elements of $\mathrm{Lie}(\xi)$, that is, elements which can be written as a linear combination of brackets having the same l-degree. Given a nonnegative $k \in \mathbb{R}$, we will denote by $\mathcal{V}_k$ the subspace of $\mathrm{Lie}_0(\xi)$ generated by all brackets having l-degree not greater than k. We denote by $\mathcal{B}_{\mathrm{S}}$ the set of elements of $\mathcal{B}$ that remain unchanged whenever the indeterminate ξ_i is interchanged with ξ_j for every $i, j \in \mathbb{I}_q$ such that $l_i = l_j$. Finally, the set $\mathcal{B}_{\mathrm{S}}^*$ of l-obstructions is the smallest Lie algebra containing $\mathcal{B}_{\mathrm{S}}$ that is closed under Lie brackets with ξ_0. Equivalently, $\mathcal{B}_{\mathrm{S}}^* \subset \mathrm{Lie}_0(\xi)$ is the Lie algebra generated by all elements of the form $\mathrm{ad}_{\xi_0}^n B$, where $n \geq 0$ and $B \in \mathcal{B}_{\mathrm{S}}$.

Next, consider a sequence $\mathbf{h} = \{h_0, h_1, \ldots, h_q\}$ of $1+q$ vector fields on $\mathcal{N}$. Given $B \in \mathrm{Lie}(\xi)$, $\mathrm{Ev}^{\mathbf{h}}(B)$ will denote the vector field on $\mathcal{N}$ obtained by substituting each occurrence of the indeterminate ξ_i in B with the vector field h_i for every $i \in \{0\} \cup \mathbb{I}_q$, and evaluating the resulting expression by interpreting the Lie bracket on $\mathrm{Lie}(\xi)$ as the Lie bracket of vector fields on $\mathcal{N}$. Further, if $p \in \mathcal{N}$, then $\mathrm{Ev}_p^{\mathbf{h}}(B)$ denotes the tangent vector in $\mathrm{T}_p\mathcal{N}$ obtained by evaluating the vector field $\mathrm{Ev}^{\mathbf{h}}(B)$ at p. Similarly, for each $k \geq 0$ and $p \in \mathcal{N}$, $\mathcal{V}_k^{\mathbf{h}}(p) \subseteq \mathrm{T}_p\mathcal{N}$ denotes the subspace $\{\mathrm{Ev}_p^{\mathbf{h}}(B) : B \in \mathcal{V}_k\}$. An l-homogeneous element $B \in \mathcal{B}_{\mathrm{S}}^*$ is said to be $\mathbf{h}$-l-neutralized at $p \in \mathcal{N}$ if $\mathrm{Ev}_p^{\mathbf{h}}(B) \in \mathcal{V}_k^{\mathbf{h}}(p)$ for some k that is strictly less than the l-degree of B.

It was shown in [6] that the system (2.8)-(2.10) is strongly accessible on $\mathcal{N}$ for every $\mu \in \mathbb{R}^3$. Hence Proposition 5.1 of [8] (equivalently, Theorem 7.20 of [11]) implies that the dynamics (2.8)-(2.10) are STLC at $p \in \mathcal{E}_\mu$ under the input constraint (2.11) if there exists a nonnegative k and an admissible weight vector l such that every l-homogeneous element of $\mathcal{B}_{\mathrm{S}}^*$ having l-degree not greater than k is $\mathbf{h}$-l-neutralized at p, and $\mathcal{V}_k^{\mathbf{h}}(p) = \mathrm{T}_p\mathcal{N}$, with $\mathbf{h} = \{f_\mu, g_1, \ldots, g_q\}$. We will use the condition just stated in the proof of Theorem 4.1. We mention, however, that the condition we have stated is only a sufficient condition for the hypotheses of the results of [8] to hold, since the results of [8] allow the use of different weight vectors to neutralize different obstructions. On the other hand, our condition is weaker than the condition stated in Theorem 1.1 of [7], which requires obstructions of all degrees to be neutralized. This latter condition, though more widely used in applications (see, for instance, [19]), is too strong to be applicable in our case. Finally, we emphasize that the condition is structural in the sense that it does not involve the actual values of the positive constants ρ_i appearing in the input constraints (2.11).

Proof of Theorem 4.1: First, suppose (4.1) holds, and let $\mathbf{h} = \{f_\mu, g_1, \ldots, g_q\}$. In this case, $\|\nu(\theta)\| = \|R^{\mathrm{T}}\mu\| = 0$. Choose $l_i = 1$ for all $i \in \{0\} \cup \mathbb{I}_q$. The two lowest possible l-degrees for a bad bracket are 3 and 5. Hence, l-homogeneous elements of

$\mathcal{B}_{\mathrm{S}}^*$ having the three lowest possible l-degrees are of the form B, $\mathrm{ad}_{\xi_0}B$, $\mathrm{ad}_{\xi_0}^2 B$, and C, where B and C are l-homogeneous elements of $\mathcal{B}_{\mathrm{S}}$ of degrees 3 and 5, respectively.

Bad brackets having l-degree 3 are of the form $[\xi_i,[\xi_0,\xi_i]]$ for some $i \in \mathbb{I}_q$. Hence the l-homogeneous element of $\mathcal{B}_{\mathrm{S}}$ having the least l-degree 3 is $B = \sum_{i=1}^q [\xi_i,[\xi_0,\xi_i]]$. Expression (3.2) along with $0 = \nu(\theta) = \sum_{i=1}^q \nu_i(\theta_i)$ allows us to compute $\mathrm{Ev}_p^{\mathbf{h}}(B) = 0$. Since $\mathrm{Ev}_p^{\mathbf{h}}(\xi_0) = f_\mu(p) = 0$, the elements $\mathrm{ad}_{\xi_0}B$ and $\mathrm{ad}_{\xi_0}^2 B$ of $\mathcal{B}_{\mathrm{S}}^*$ also evaluate to 0 at p.

There are only two types of bad brackets of the next lowest degree 5. A bad bracket C_{ij} of the first type has $|C_{ij}|_0 = 1$ and $|C_{ij}|_i = |C_{ij}|_j = 2$ for some $i,j \in \mathbb{I}_q$. One can use (3.1) and (3.2) to show that $\mathrm{Ev}^{\mathbf{h}}(C_{ij}) = 0$ if $i \neq j$. Given $i \in \mathbb{I}_q$, $\mathrm{Ev}_p^{\mathbf{h}}(C_{ii})$ equals, to within a sign change, the tangent vector $[g_i,[g_i,[g_i,[f_\mu,g_i]]]](p)$, whose expression can be easily calculated from (3.2) to be $(R(J^{-1}\nu_i(\theta_i))^\times, 0)$. Since $\sum_{i=1}^q \nu_i(\theta_i) = \nu(\theta) = 0$, the l-homogeneous element $C = \sum_{i=1}^q C_{ii}$ of $\mathcal{B}_{\mathrm{S}}$ evaluates to 0 under $\mathrm{Ev}_p^{\mathbf{h}}$. A bad bracket C_i of the second type has $|C_i|_0 = 3$ and $|C_i|_i = 2$ for some $i \in \mathbb{I}_q$. For such a bracket, $\mathrm{Ev}^{\mathbf{h}}(C_i)$ can be, to within a sign change, one of $c_1 = [\mathrm{ad}_{f_\mu}^1 g_i, \mathrm{ad}_{f_\mu}^2 g_i]$, $c_2 = [f_\mu,[g_i,\mathrm{ad}_{f_\mu}^2 g_i]]$ or $c_3 = [g_i, \mathrm{ad}_{f_\mu}^3 g_i]$. We have $c_1(p) = 0$ from (3.7). Letting $\mu = 0$ in (3.6) gives $[g_i, \mathrm{ad}_{f_\mu}^2 g_i]](p) = 0$. Since $f_\mu(p) = 0$ also, it follows that $c_2(p) = 0$. The Jacobi identity for vector fields implies that $c_3(p) = 0$ as well.

It follows that every l-homogeneous element of $\mathcal{B}_{\mathrm{S}}^*$ having l-degree not greater than 5 evaluates to 0 at p, and is thus $\mathbf{h}$-l-neutralized. The space $\mathcal{V}_5^{\mathbf{h}}(p)$ contains the $3+q$ tangent vectors obtained by evaluating the vector fields $[f_\mu,g_1]$, $[g_1,[f_\mu,g_1]]$, $[[g_1,[f_\mu,g_1]],[f_\mu,g_1]]$, and g_i, $i \in \mathbb{I}_q$, at p. Since the vectors $J^{-1}\nu_1(\theta_1)$ and $J^{-1}\nu_1'(\theta_1)$ are linearly independent, it follows that $\mathcal{V}_5^{\mathbf{h}}(p) = \mathrm{T}_p\mathcal{N}$. Hence STLC at p follows.

Before continuing, we claim that if either (4.2) or (4.3) holds, then there exists an index set $I \subseteq \mathbb{I}_q$ and positive constants $\lambda_i > 0$, $i \in I$, such that

$$\nu(\theta)^{\mathrm{T}} \left[\sum_{i \in I} \lambda_i \nu_i(\theta_i) \right] = 0, \tag{4.4}$$

and either $\mathrm{span}\{\nu_i'(\theta_i) : i \in I\}$ has dimension 2, or there exists $i \in I$ such that $\nu_i^{\mathrm{T}}(\theta_i)\nu(\theta) \neq 0$. Indeed, suppose (4.2) holds. Since $0 \leq \nu(\theta)^{\mathrm{T}}\nu(\theta) = \sum_{i=1}^q \nu(\theta)^{\mathrm{T}}\nu_i(\theta_i)$, it follows that there exist distinct i, $j \in \mathbb{I}_q$ such that $\nu(\theta)^{\mathrm{T}}\nu_i(\theta_i) < 0 < \nu(\theta)^{\mathrm{T}}\nu_j(\theta_j)$. Our claim then holds by letting $I = \{i,j\}$, $\lambda_i = \nu(\theta)^{\mathrm{T}}\nu_j(\theta_j)$ and $\lambda_j = -\nu(\theta)^{\mathrm{T}}\nu_i(\theta_i)$. On the other hand, if (4.3) holds, then our claim holds by letting $I = \{i : \nu(\theta)^{\mathrm{T}}\nu_i(\theta_i) = 0\}$ and $\lambda_i = 1$, $i \in I$.

Next, assume that either (4.2) or (4.3) holds, and let $I \subseteq \mathbb{I}_q$ be as guaranteed by our claim above. We need only consider the case $\nu(\theta) \neq 0 \neq \mu$. Consider the input scaling $u_i = \sqrt{\lambda_i}\hat{u}_i$, $i \in I$, $u_i = \hat{u}_i$, $i \notin I$. We may rewrite (2.8) as

$$\dot{y}(t) = f_\mu(y(t)) + \hat{g}_1(y(t))\hat{u}_1(t) + \cdots + \hat{g}_q(y(t))\hat{u}_q(t), \tag{4.5}$$

where $\hat{g}_i$ equals $\sqrt{\lambda_i}g_i$ if $i \in I$ and equals g_i if $i \notin I$. Clearly, (2.8) is STLC subject to the input constraint (2.11) if and only if (4.5) is STLC subject to the input constraint $\hat{u} \in \mathcal{H}_{\hat{\rho}}$ for a suitably defined positive vector $\hat{\rho}$. Hence, it suffices to show that (4.5) satisfies the sufficient condition given by Proposition 5.1 of [8]. For this purpose, we redefine $\mathbf{h} = \{f_\mu, \hat{g}_1, \ldots, \hat{g}_q\}$.

Let $l_0 = 1$, $l_i = 1.5$ for $i \in I$, and $l_i = 2.5$ for $i \notin \{0\} \cup I$. The two lowest possible l-degrees for a bad bracket are 4 and 6. As a result, each l homogeneous element of $\mathcal{B}_{\rm S}^*$ having l-degree not greater than 5 is of one of the forms B or $\mathrm{ad}_{\xi_0}B$, where $B \in \mathcal{B}_{\rm S}$ has l-degree 4. Bad brackets having l-degree 4 are of the form $[\xi_i, [\xi_0, \xi_i]]$ for some $i \in I$. Hence the l-homogeneous element of $\mathcal{B}_{\rm S}$ having the least l-degree 4 is $B = \sum_{i\in I}[\xi_i, [\xi_0, \xi_i]]$. We have $\mathrm{Ev}^{\mathbf{h}}(B) = \sum_{i\in I}[\hat{g}_i[f_\mu, \hat{g}_i]] = \sum_{i\in I} \lambda_i[g_i, [f_\mu, g_i]]$. We also have $\mathrm{Ev}_p^{\mathbf{h}}(\mathrm{ad}_{\xi_0}B) = \sum_{i\in I} \lambda_i[f_\mu, [g_i, [f_\mu, g_i]]](p) = \sum_{i\in I} \lambda_i[g_i, \mathrm{ad}^2_{f_\mu} g_i](p)$, where the last equality uses the Jacobi identity. The condition (4.4) along with (3.2) and (3.6) implies that each of the tangent vectors $\mathrm{Ev}_p^{\mathbf{h}}(B)$ and $\mathrm{Ev}_p^{\mathbf{h}}(\mathrm{ad}_{\xi_0}B)$ is of the form $(R(J^{-1}u)^\times, 0)$, where u is orthogonal to $R^{\rm T}\mu = \nu(\theta)$. Consequently, u is contained in the span of the linearly independent vectors $\nu_i'(\theta_i)$ and $J^{-1}\nu_i'(\theta_i) \times R^{\rm T}\mu$, for any arbitrarily chosen $i \in I$. Therefore, the expressions (3.1) and (3.5) show that $\mathrm{Ev}_p^{\mathbf{h}}(B)$ and $\mathrm{Ev}_p^{\mathbf{h}}(\mathrm{ad}_{\xi_0}B)$ are contained in the span of the tangent vectors $[f_\mu, \hat{g}_i](p)$ and $[f_\mu, [f_\mu, \hat{g}_i]](p)$, both of which are contained in $\mathcal{V}_{3.5}^{\mathbf{h}}(p)$. Hence, all l-homogeneous elements of $\mathcal{B}_{\rm S}^*$ having l-degree not greater than 5 are $\mathbf{h}$-l-neutralized.

If the dimension of $\mathrm{span}\{\nu_i'(\theta_i) : i \in I\}$ is 2, then there exists $i, j \in I$ such that $\nu_i'(\theta_i)$ and $\nu_j'(\theta_j)$ are linearly independent. In this case, the subspace $\mathcal{V}_5^{\mathbf{h}}(p)$ contains the $3 + q$ tangent vectors $[f_\mu, \hat{g}_i](p)$, $[f_\mu, \hat{g}_j](p)$, $[[f_\mu, \hat{g}_i], [f_\mu, \hat{g}_j]](p)$ and $\hat{g}_k$, $k \in \mathbb{I}_q$, which are linearly independent by (3.1) and (3.3). On the other hand, if $i \in I$ is such that $\nu_i^{\rm T}(\theta_i)\nu(\theta) \neq 0$, then a simple computation of a scalar triple product shows that the vectors $\nu_i'(\theta_i)$, $J^{-1}\nu_i'(\theta_i) \times \nu(\theta)$ and $\nu_i(\theta_i)$ are linearly independent. Hence, in this case, the space $\mathcal{V}_5^{\mathbf{h}}(p)$ contains the $3+q$ linearly independent tangent vectors $[f_\mu, \hat{g}_i](p)$, $[f_\mu, [f_\mu, \hat{g}_i]](p)$, $[\hat{g}_i, [f_\mu, \hat{g}_i]](p)$ and $\hat{g}_k(p)$, $k \in \mathbb{I}_q$. In either case, $\mathcal{V}_5^{\mathbf{h}}(p) = \mathrm{T}_p\mathcal{N}$, and STLC follows. $\square$

Example 4.2. Consider a spacecraft carrying the CMG array described in Example 2.1. Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in \mathcal{E}_\mu$. If $\theta = \theta^{\rm C}$ introduced in Example 2.1, then (4.1) holds. On the other hand, if $\theta = \theta^{\rm D}$, then (4.2) holds. In either case, Theorem 4.1 implies that the dynamics (2.8)-(2.10) are STLC at p under the input constraint (2.11). Finally, if $\theta = \theta^{\rm E}$, then none of the conditions (4.1)-(4.3) hold, and Theorem 4.1 cannot be applied. Our main result of the next section will allow us to determine the STLC at equilibria involving the configuration $\theta^{\rm E}$.

5. Necessary Conditions for STLC at Critically Singular Configurations. In this section, we will use Theorem 1.2 of [22] to show that STLC fails to hold in one of the cases not covered by Theorem 4.1. First, we note that Theorem 1.2 of [22] gives a sufficient condition for an uncontrolled reference trajectory $y(\cdot)$ of (2.8) to lie on the boundary of the reachable set from the initial state $y(0)$ of the reference trajectory for some sufficiently small length of time. To apply the sufficient condition, we will choose the reference trajectory to be the constant trajectory starting from an uncontrolled equilibrium point $p \in \mathcal{N}$ of (2.8), so that, if the sufficient condition holds, it will allow us to conclude that p lies on the boundary of the set of states reachable from p in sufficiently small time, thus ruling out STLC at p. Since the result of [22] involves the adjoint system corresponding to the uncontrolled system

$$\dot{y}(t) = f_\mu(y(t)), \tag{5.1}$$

we will briefly introduce the notation necessary to investigate the adjoint system of (5.1) without performing cumbersome coordinate-based calculations.

The adjoint system of (5.1) is the Hamiltonian system on the cotangent bundle $\mathrm{T}^*\mathcal{N}$ (together with its canonical symplectic structure [1, Sec. 3.2]) whose Hamiltonian function $H_\mu : \mathrm{T}^*\mathcal{N} \to \mathbb{R}$ is given by $H_\mu(\Lambda) = \Lambda(f_\mu(\pi^*(\Lambda)))$ for every $\Lambda \in \mathrm{T}^*\mathcal{N}$, where $\pi^* : \mathrm{T}^*\mathcal{N} \to \mathcal{N}$ is the canonical cotangent bundle projection.

The cotangent bundle $\mathrm{T}^*\mathcal{N}$ may be identified with $\mathrm{T}^*\mathrm{SO}(3) \times \mathrm{T}^*\mathbb{T}^q$ in a natural way. Since SO(3) is a Lie group, we may use left translations to identify $\mathrm{TSO}(3) = \mathrm{SO}(3) \times \mathrm{so}(3)$ and $\mathrm{T}^*\mathrm{SO}(3) = \mathrm{SO}(3) \times \mathrm{so}(3)^*$, where $\mathrm{so}(3)^*$ is the vector space dual to so(3) [1, Sec. 4.4]. Using the inner product $\langle v_1^\times, v_2^\times \rangle \stackrel{\text{def}}{=} \frac{1}{2}\mathrm{trace}[(v_1^\times)^{\mathrm{T}} v_2^\times] = v_1^{\mathrm{T}} v_2$ on so(3), we may identify $\mathrm{so}(3)^*$ with so(3), and hence $\mathrm{T}^*\mathrm{SO}(3)$ with TSO(3). The cotangent and tangent bundles of $\mathbb{T}^q$ may also be canonically identified with $\mathbb{T}^q \times \mathbb{R}^q$. Thus, an element Λ of $\mathrm{T}^*\mathcal{N}$ may be identified with the 4-tuple $(R, \lambda_R^\times, \theta, \lambda_\theta)$, where $(R, \theta) = \pi^*(\Lambda) \in \mathcal{N}$, $\lambda_R \in \mathbb{R}^3$, and $\lambda_\theta \in \mathbb{R}^q$. Similarly, an element v of $\mathrm{T}\mathcal{N}$ may be identified with the 4-tuple $(R, v_R^\times, \theta, v_\theta)$, where $(R, \theta) = \pi(v) \in \mathcal{N}$, $v_R \in \mathbb{R}^3$, and $v_\theta \in \mathbb{R}^q$, with $\pi : \mathrm{T}\mathcal{N} \to \mathcal{N}$ being the canonical tangent bundle projection. Thus, in the notation above, if $\Lambda \in \mathrm{T}^*\mathcal{N}$ and $v \in \mathrm{T}\mathcal{N}$ are such that $\pi^*(\Lambda) = \pi(v)$, then

$$\Lambda(v) = \lambda_R^{\mathrm{T}} v_R + \lambda_\theta^{\mathrm{T}} v_\theta. \tag{5.2}$$

Hence, the Hamiltonian function of the adjoint system of (5.1) may be written as

$$H_\mu(\Lambda) = \lambda_R^{\mathrm{T}} J^{-1}(R^{\mathrm{T}}\mu - \nu(\theta)),\ \Lambda = (R, \lambda_R^\times, \theta, \lambda_\theta) \in \mathrm{T}^*\mathcal{N}. \tag{5.3}$$

Our next lemma gives an equilibrium point of the adjoint system of (5.1).

Lemma 5.1. *Suppose* $\mu \neq 0$, *and* $(R, \theta) \in \mathcal{E}_\mu$ *is such that* $\theta \in \mathcal{C}$. *Then* $\Lambda^* \stackrel{\text{def}}{=} (R, (J\nu(\theta))^\times, \theta, 0) \in \mathrm{T}^*_{(R,\theta)}\mathcal{N}$ *is an equilibrium point of the adjoint system of (5.1).*

Proof. Every critical point of a Hamiltonian function is an equilibrium point of the corresponding Hamiltonian system [1, Prop. 3.4.16]. Hence it suffices to show that Λ^* is a critical point of the Hamiltonian function (5.3). Consider the curve $\gamma : [0, 1] \to \mathrm{T}^*\mathcal{N}$ given by $\gamma(t) = (Re^{tw_1^\times}, (J\nu(\theta) + w_2 t)^\times, \theta + w_3 t, w_4 t)$, where $w_1, w_2 \in \mathbb{R}^3$ and $w_3, w_4 \in \mathbb{R}^q$ are arbitrary. Note that $\gamma(0) = \Lambda^*$, and $\dot{\gamma}(0)$ represents an arbitrary vector tangent to $\mathrm{T}^*\mathcal{N}$ at Λ^*. Using (5.3), we may easily compute

$$H_\mu(\gamma(t)) = [\nu^{\mathrm{T}}(\theta) + t w_2^{\mathrm{T}} J^{-1}][e^{-tw_1^\times} R^{\mathrm{T}}\mu - \nu(\theta + t w_3)],\ t \in [0, 1], \tag{5.4}$$

so that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H_\mu(\gamma(t)) = \nu^{\mathrm{T}}(\theta) \left[-w_1 \times R^{\mathrm{T}}\mu - \sum_{i=1}^{q} \nu_i'(\theta_i) w_3^i \right] + w_2^{\mathrm{T}} J^{-1}[R^{\mathrm{T}}\mu - \nu(\theta)], \tag{5.5}$$

where w_3^i denotes the ith component of w_3. On recalling that $(R, \theta) \in \mathcal{E}_\mu$ and $\theta \in \mathcal{C}$, so that $R^{\mathrm{T}}\mu = \nu(\theta)$ and $\nu^{\mathrm{T}}(\theta)\nu_i'(\theta_i) = 0$ for all $i \in \mathbb{I}_q$, we immediately conclude that the righthandside of (5.5) is zero. Since the choice of the representative curve γ was arbitrary, we conclude that Λ^* is a critical point of the Hamiltonian function of the adjoint system of (5.1). This completes the proof. ∎

Remark 5.2. One may use (5.2) along with Lemma 3.2 to show that, if $\theta \in \mathcal{C}$ and $p = (R, \theta) \in \mathcal{E}_\mu$ for some $\mu \neq 0$, then $\Lambda^*(\mathrm{ad}_{f_\mu}^k g_i(p)) = 0$ for all $n \in \{0\} \cup \mathbb{I}_{q+2}$ and $i \in \mathbb{I}_q$. Thus, Λ^* represents a left null-vector of the controllability matrix of the linearization at p of (2.8) in some local coordinates on $\mathcal{N}$. Indeed, computations in suitably chosen local coordinates show that 0 is an uncontrollable eigenvalue of the linearization of (2.8) at p, and Λ^* is related to the corresponding left eigenvector of the state matrix of the linearization.

Our next result is the main result of this section.

Theorem 5.3. *Suppose* $\mu \neq 0$. *Let* $\theta \in \mathcal{C}$ *be such that*

$$\min\{\nu_i(\theta_i)^{\mathrm{T}}\nu(\theta) : 1 \leq i \leq q\} > 0. \tag{5.6}$$

If $p = (R, \theta) \in \mathcal{E}_\mu$, then the dynamics described by (2.8)-(2.10) are not STLC at p under the input constraint (2.11).

Proof. Let $T > 0$, and consider the reference trajectory $y(t) \equiv p$ of the uncontrolled system (5.1) on the interval $[0, T]$. Let Λ^* be as in Lemma 5.1, and consider the constant curve $\gamma(t) \equiv \Lambda^*$ in $\mathrm{T}^*\mathcal{N}$ on the interval $[0, T]$. We trivially have $\gamma(t) \in \mathrm{T}^*_{y(t)}\mathcal{N}$ for all $t \in [0, T]$. Moreover, by Lemma 5.1, γ is a (constant) trajectory of the adjoint system of (5.1).

We may use (5.2) along with Lemma 3.2 to show that, for all $n \geq 0$, $i \in \mathbb{I}_q$ and $t \in [0, T]$, $\gamma(t)(\mathrm{ad}^k_{f_\mu} g_i(p)) = \Lambda^*(\mathrm{ad}^k_{f_\mu} g_i(p)) = 0$. Moreover, the vector fields $\mathrm{ad}^k_{f_\mu}[g_i, g_j]$ are identically zero. Hence the condition i) of Theorem 1.2 of [22] is satisfied.

To check condition ii) of the same result, consider the matrix $L \in \mathbb{R}^{q \times q}$ defined by $L_{ij} = \gamma(0)([\mathrm{ad}_{f_\mu} g_i, g_j](y(0)))$. The expressions (3.2) and (5.2) allow us to compute L as the diagonal matrix such that $L_{ii} = \nu^{\mathrm{T}}(\theta)\nu_i(\theta_i)$ for each $i \in \mathbb{I}_q$. The assumption (5.6) implies that L is positive definite. Hence condition ii) of Theorem 1.2 of [22] is satisfied. Finally, condition iii) of Theorem 1.2 of [22] holds trivially, since all brackets of the form $[g_i, [g_j, g_k]]$ are identically zero. Hence Theorem 1.2 of [22] implies that there exists $\hat{T} \in (0, T]$ such that $y(t) = p$ lies on the boundary of the reachable set at t from $y(0) = p$ for all $t \in [0, \hat{T}]$. This completes the proof. □

Note that for $\theta \in \mathcal{C}$, (5.6) is equivalent to the condition that θ is an external singularity in the sense defined earlier. Thus Theorem 5.3 states that if the critically singular CMG configuration at the equilibrium p is an external singularity, then there exist terminal states arbitrarily close to p and a time duration $T > 0$ such that any set of gimbal motions satisfying the rate constraint (2.11) takes at least an amount of time T to steer the trajectory of (2.8)-(2.10) from p to any one of these terminal states.

Critically singular configurations are critical points of the function η, and include as a special case local maximizers of η. The following corollary of Theorem 5.3 shows that STLC fails at equilibria in which the CMG configuration is a local maximizer of the total CMG angular momentum magnitude η. The proof involves showing that a local maximizer of the CMG angular momentum magnitude is an external singularity.

Corollary 5.4. *Suppose $\theta \in \mathcal{C}$ is a local maximizer for the function η, and suppose $p = (R, \theta) \in \mathcal{E}_\mu$ for some $\mu \in \mathbb{R}^3$, $\mu \neq 0$. Then the spacecraft attitude dynamics described by (2.8)-(2.10) are not STLC at p under the input constraint (2.11).*

Proof. Since $\theta \in \mathcal{C}$ is a local maximizer for the smooth function η, the $q \times q$ Hessian matrix of η at θ is a negative semidefinite matrix. In particular, the diagonal elements of the Hessian are nonpositive. The ith diagonal element of the Hessian of η at θ is easily seen to be $\frac{\partial^2 \eta}{\partial \theta_i^2}(\theta) = 2\|\nu_i'(\theta_i)\|^2 - 2\nu^{\mathrm{T}}(\theta)\nu_i(\theta_i)$. Applying the fact that the diagonal elements of the Hessian are nonpositive leads to $\nu^{\mathrm{T}}(\theta)\nu_i(\theta_i) \geq \|\nu_i'(\theta_i)\|^2 > 0$ for all $i \in \mathbb{I}_q$. Thus (5.6) holds, and the result follows from Theorem 5.3. □

As another corollary of Theorem 5.3, we prove that STLC fails at all equilibria if the spacecraft carries only one CMG.

Corollary 5.5. *Suppose $q = 1$. Let $\mu \in \mathbb{R}^3$ and let $p = (R, \theta) \in \mathcal{E}_\mu$. Then, the spacecraft attitude dynamics described by (2.8)-(2.10) are not STLC at p under the input constraint (2.11).*

Proof. In the case $q = 1$, the function η is a constant function. As a result θ is a local maximizer of the function η, and the conclusion follows from Corollary 5.4. □

Example 5.6. Continuing further with the CMG array introduced in Example

2.1, let $\mu \in \mathbb{R}^3$ and $R \in \mathrm{SO}(3)$ be such that $p \stackrel{\text{def}}{=} (R, \theta^{\mathrm{E}}) \in \mathcal{E}_\mu$. Then (5.6) holds, and hence Theorem 5.3 implies that the dynamics (2.8)-(2.10) are not STLC at p.

Before closing the section, we remark that the conditions of theorems 4.1 and 5.3 do not cover the case where $\nu(\theta) \neq 0$ and

$$\min\{\nu_i(\theta_i)^{\mathrm{T}}\nu(\theta) : 1 \leq i \leq q\} = 0, \ \dim \operatorname{span}\{\nu_i'(\theta_i) : \nu_i(\theta_i)^{\mathrm{T}}\nu(\theta) = 0\} = 1. \qquad (5.7)$$

Physically, this case represents a situation where k of the q CMGs have a common gimbal axis along the total angular momentum vector $\nu(\theta)$ and have colinear individual angular momentum vectors, while the angular momenta of the remaing CMGs have a positive component along the nonzero vector $\nu(\theta)$. It is easy to conceive of a CMG array possessing such a critically singular configuration. Hence, even though theorems 3.3 and 5.3 enable us to determine STLC or its lack for a large class of singular configurations, the problem of determining STLC in the presence of singular configurations still remains partially open.

6. Stabilizability. In this section, we analyze the feedback stabilizability of equilibria of (2.8). Our first result below asserts that equilibrium points of (2.8) in which the CMG array is not in a critically singular configuration is locally asymptotically stabilizable using smooth static feedback.

THEOREM 6.1. *Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in \mathcal{E}_\mu$ is such that $\theta \notin \mathcal{C}$. Then p is locally asymptotically stabilizable by smooth static state feedback.*

Proof. Theorem 3.3 implies that the linearization of (2.8) at p is controllable (see [13]), and hence stabilizable. It now follows that (2.8) is locally asymptotically stabilizable at p by smooth static state feedback [27, Prop. 1]. □

Our next result shows that equilibrium points at which the magnitude of the total angular momentum of the CMG array either has a local maximum or a nonzero local minimum, cannot be stabilized using continuous feedback. Note that, in both cases, the CMG array is in a critically singular configuration. The proof is based on showing that Brockett's necessary condition [27, 10] for continuous stabilizability is violated.

THEOREM 6.2. *Let $\mu \in \mathbb{R}^3$, and suppose $p = (R, \theta) \in \mathcal{E}_\mu$ is such that $\nu(\theta) \neq 0$, and θ is a either a local maximizer or a local minimizer of the function η. Then p is not locally asymptotically stabilizable through continuous time-invariant feedback.*

Proof. Since $\nu(\theta)^{\mathrm{T}}(\hat{R}^{\mathrm{T}}\mu) > 0$ and $\nu(\theta)^{\mathrm{T}}\nu(\hat{\theta}) > 0$ hold for $(\hat{R}, \hat{\theta}) = (R, \theta)$, we may choose an arbitrarily small open neighbourhood $\mathcal{V} \subseteq \mathcal{N}$ of p such that $\nu(\theta)^{\mathrm{T}}(\hat{R}^{\mathrm{T}}\mu) > 0$ and $\nu(\theta)^{\mathrm{T}}\nu(\hat{\theta}) > 0$ for all $(\hat{R}, \hat{\theta}) \in \mathcal{V}$. Suppose p is locally asymptotically stabilizable through continuous time-invariant feedback. Brockett's necessary condition for continuous stabilizability [27, Thm. 3], [10, Thm. 1], when applied to (2.8) implies that the image of $\mathcal{V}$ under the map $h : (\hat{R}, \hat{\theta}) \mapsto \hat{R}^{\mathrm{T}}\mu - \nu(\hat{\theta})$ contains an open neighbourhood of 0. Hence there exists $\delta > 0$ such that, for every $\epsilon \in (-\delta, \delta)$, the vector $\epsilon\nu(\theta)$ is contained in $h(\mathcal{V})$.

First, pick $\epsilon \in (-\delta, 0)$, and choose $(\hat{R}, \hat{\theta}) \in \mathcal{V}$ such that $h(\hat{R}, \hat{\theta}) = \hat{R}^{\mathrm{T}}\mu - \nu(\hat{\theta}) = \epsilon\nu(\theta)$. Since $p \in \mathcal{E}_\mu$, we have $\nu(\theta) = R^{\mathrm{T}}\mu$, so that $\|\nu(\theta)\|^2 = \|\mu\|^2 = \|\hat{R}\mu\|^2$. Therefore, we have $\|\nu(\theta)\|^2 - \|\nu(\hat{\theta})\|^2 = \|\hat{R}^{\mathrm{T}}\mu\|^2 - \|\nu(\hat{\theta})\|^2 = (\hat{R}^{\mathrm{T}}\mu + \nu(\hat{\theta}))^{\mathrm{T}}(\hat{R}^{\mathrm{T}}\mu - \nu(\hat{\theta})) = \epsilon(\hat{R}^{\mathrm{T}}\mu + \nu(\hat{\theta}))^{\mathrm{T}}\nu(\theta) < 0$. This shows that θ is not a maximizer of the map η over $\mathcal{V}$. Similarly, by choosing $\epsilon \in (0, \delta)$, it can be shown that θ is not a minimizer of η over $\mathcal{V}$. Since $\mathcal{V}$ can be chosen to be arbitrarily small, it follows that θ is neither a local maximizer nor a local minimizer for η, and the result follows. □

As a corollary of the above result, we show that no equilibrium of a spacecraft actuated by a single CMG is stabilizable through continuous feedback.

Corollary 6.3. *Suppose $q = 1$ and let $\mu \in \mathbb{R}^3$. Suppose $p = (R, \theta) \in \mathcal{E}_\mu$. Then p is not locally asymptotically stabilizable through continuous time-invariant feedback.*

Proof. The result follows from Theorem 6.2 by noting that, for a single CMG, the function η is a constant function, and hence every configuration is a local maximizer of the function η. ∎

Example 6.4. Consider, once again, the CMG pyramid array described in Example 2.1. Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in \mathcal{E}_\mu$. Theorem 6.1 implies that p is locally asymptotically stabilizable through smooth feedback if $\theta \in \{\theta^{\mathrm{A}}, \theta^{\mathrm{B}}\}$. On the other hand, Theorem 6.2 implies that p is not locally asymptotically stabilizable through continuous feedback if $\theta = \theta^{\mathrm{E}}$ and $\sin^2 \alpha > \frac{1}{2}$.

7. Conclusions. The purpose of this paper was to investigate in what manner the presence of singular CMG configurations affects small-time local controllability and stabilizability of the attitude dynamics of a spacecraft actuated by a CMG array. Our main achievement has been to identify a class of singular CMG configurations called critically singular configurations, at which proving local controllability and stabilizability is problematic. The dynamics near those equilibria in which the configuration of the CMG array is possibly singular, but not critically singular, are linearly controllable and hence small-time locally controllable and stabilizable on an angular momentum level set. In contrast, for equilibria in which the configuration of the CMG array is critically singular, STLC holds only under additional assumptions on the CMG configuration. Specifically, STLC fails if the critically singular CMG configuration is an external singularity in the sense that we have defined. In particular, STLC and stabilizability fail at equilibria at which the magnitude of the total angular momentum of the CMG array has a local maximum. The problem of determining STLC still remains open for certain critically singular configurations, while the problem of determining stabilizability remains open for a large class of critically singular configurations.

Appendix A. Proofs of lemmas.

Proof of Lemma 3.1. To arrive at a contradiction, suppose that $\nu_i'(\theta_i) \times \nu_j'(\theta_j) = 0$ for all $i, j \in \mathbb{I}_q$. It follows that $\nu_i(\theta_i)^{\mathrm{T}} \nu_j'(\theta_j) = 0$ for all $i, j \in \mathbb{I}_q$. Hence, $\nu(\theta)^{\mathrm{T}} \nu_j'(\theta_j) = 0$ for all $j \in \mathbb{I}_q$. Therefore, $\theta \in \mathcal{C}$, which is a contradiction. The contradiction implies that, for every $k \in \mathbb{I}_q$, there exists an $n \in \mathbb{I}_q$ such that $\nu_k'(\theta_k) \times \nu_n'(\theta_n) \neq 0$. In particular, the vectors $\{\nu_1'(\theta_1), \ldots, \nu_m'(\theta_q)\}$ span the two-dimensional subspace orthogonal to the vector v.

Since $\theta \in \mathcal{S}_v \backslash \mathcal{C}$, it follows that $\nu(\theta) \neq 0$ and $v \times \nu(\theta) \neq 0$. To arrive at a contradiction, suppose that $v^{\mathrm{T}}(J^{-1}\nu_i'(\theta_i) \times \nu(\theta)) = 0$ for all $i \in \mathbb{I}_q$. Using basic properties of the scalar triple product and noting that $J = J^{\mathrm{T}}$, it follows that $\nu_i'(\theta_i)^{\mathrm{T}}(J^{-1}(v \times \nu(\theta))) = 0$ for all $i \in \mathbb{I}_q$. Thus the vector $J^{-1}(v \times \nu(\theta))$ is orthogonal to every vector in the two-dimensional subspace orthogonal to v. Hence there exists $\gamma \in \mathbb{R}\backslash\{0\}$ such that $v \times \nu(\theta) = \gamma J v$. This implies that $v^{\mathrm{T}} J v = 0$, which contradicts the fact that J is positive definite. The contradiction implies that there exists $i \in \mathbb{I}_q$ such that $v^{\mathrm{T}}(J^{-1}\nu_i'(\theta_i) \times \nu(\theta)) \neq 0$. As shown in the previous paragraph, there exists $j \in \mathbb{I}_q$ such that $\nu_i'(\theta_i) \times \nu_j'(\theta_j) \neq 0$. □

Proof of Lemma 3.2. Choose $i \in \mathbb{I}_q$. First, assume $\mu \neq 0$. Then it follows that $\nu(\theta) \neq 0$ and $\theta \in \mathcal{S}_v$ for $v = \|\nu(\theta)\|^{-1}\nu(\theta)$. We claim that, for every $n \geq 1$, there exist functions $h_{n,i} : \mathcal{N} \to \mathbb{R}^3$ and $c_{n,i} : \mathcal{N} \to \mathbb{R}^3$ satisfying $\nu(\theta)^{\mathrm{T}} h_{n,i}(p) = 0$ and $c_{n,i}(p) = 0$ such that, for every $x = (\hat{R}, \hat{\theta}) \in \mathcal{N}$,

$$\mathrm{ad}_{f_\mu}^n g_i(x) = (\hat{R}(J^{-1}h_{n,i}(x) + c_{n,i}(x))^\times, 0). \tag{A.1}$$

We prove this claim by induction on n.

It follows from (3.1) that the claim is true for $n = 1$ with $h_{1,i}(x) = \nu_i'(\theta_i)$ and $c_{1,i}(x) = 0$ for all $x \in \mathcal{N}$. Next, suppose the claim is true for $n = k$. Consider $x = (\hat{R}, \hat{\theta}) \in \mathcal{N}$, denote $\omega(x) = J^{-1}(\hat{R}^{\mathrm{T}}\mu - \nu(\hat{\theta}))$, and note that $\omega(p) = 0$. An algebraic computation using (2.1) yields

$$\mathrm{ad}_{f_\mu}^{k+1} g_i(x) = (\hat{R}(J^{-1}h_{k+1,i}(x) + c_{k+1,i}(x))^\times, 0), \tag{A.2}$$

where $h_{k+1,i}(x) \overset{\mathrm{def}}{=} J^{-1}h_{k,i}(x) \times \hat{R}^{\mathrm{T}}\mu$ and $c_{k+1,i}(x) \overset{\mathrm{def}}{=} \omega(x) \times (J^{-1}h_{k,i}(x) + c_{k,i}(x)) + J^{-1}(c_{k,i}(x) \times R^{\mathrm{T}}\mu) + \psi_{k+1,i}(x)$ with

$$\psi_{k+1,i}(x) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[J^{-1}(h_{k,i}(x + sf_\mu(x)) + (c_{k,i}(x + sf_\mu(x))\right].$$

Since $f_\mu(p) = 0$, it follows that $\psi_{k+1,i}(p) = 0$, and hence, by the induction hypothesis, $c_{k+1,i}(p) = 0$. Furthermore, $h_{k+1,i}(p)$ equals $J^{-1}h_{k,i}(p) \times \nu(\theta)$ and satisfies $\nu(\theta)^{\mathrm{T}}h_{k+1,i}(p) = 0$. Induction now implies that (A.1) holds for all $n \geq 1$. Thus, for every $n \geq 1$ and every $i \in \mathbb{I}_q$, the tangent vector $\mathrm{ad}_{f_\mu}^n g_i(p)$ is contained in the two-dimensional subspace $\{(R(J^{-1}w)^\times, 0) : w \in \mathbb{R}^3, w^{\mathrm{T}}\nu(\theta) = 0\}$ of $\mathrm{T}_p\mathcal{N}$. This proves the first assertion in the case $\mu \neq 0$, as well as the second assertion.

Next, assume $\mu = 0$ and $\theta \in \mathcal{C}$. Equation (3.5) implies that $\mathrm{ad}_{f_\mu}^2 g_i(p) = 0$. Since $f_\mu(p) = 0$, an easy computation shows that $\mathrm{ad}_{f_\mu}^{k+1} g_i(p) = 0$ whenever $\mathrm{ad}_{f_\mu}^k g_i(p) = 0$. Induction now implies that $\mathrm{ad}_{f_\mu}^n g_i(p) = 0$ for all $n \geq 2$. Hence $\mathrm{span}\{\mathrm{ad}_{f_\mu}^n g_i(p) : i \in \mathbb{I}_q, n \geq 1\} = \mathrm{span}\{[f_\mu, g_i](p) : i \in \mathbb{I}_q\}$, which has dimension not greater than two since $\theta \in \mathcal{C}$. This completes the proof of the first assertion. $\square$

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