An Accurate Two-Dimensional Panel Method

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Abstract

A new panel technique which produces a globally accurate velocity field has been developed. Traditional panel techniques panelise the body into a set of linear panels and distribute some kind of singularity on the surface of these panels. The singularities normally used are sources, doublets and vorticity. Such panel methods produce high accuracy at the control point but do not produce accurate results globally and as one approaches the edges of the panels the velocity field diverges. This is known as the edge effect. Most computations that use traditional panel methods are not affected by this edge effect since these panel techniques are typically used to find the velocity/pressure of the fluid on the panelised body at certain control points where the no penetration (or no tangency) boundary condition is applied and at these points there is very little error. However, there are certain applications, like particle based flow solvers, where a globally accurate velocity field is required. It can be shown that however high the order of the singularity distribution on the panels, the edge effect is due to the mismatch of the slope between two adjacent panels. Hence, the natural approach to this problem is to panelise the body using cubic panels rather than linear panels by matching the slopes at the edge of each panel. The fundamental equations for the velocity field and potential due to such a cubic panel having a linear vorticity distribution are derived in this article. Also addressed are certain issues involved in the derivation. The accuracy of the method is then demonstrated by comparing the results with that obtained by traditional methods.

1. Introduction

Panel methods offer a very elegant and powerful means of computation for flow past arbitrary bodies in two and three dimensions under various conditions of flow. The power of the method is both due to the fact that the differential equations are reduced to an integral form along the surface of the body and because the body in question is directly represented by a distribution of singularities on its surface. Hess and Smith [1] laid the foundation for the source panel method. The idea of the vortex panel method is due to Martensen [4] and is extended by Lewis [3]. Katz and Plotkin [2] give a comprehensive overview of panel methods in general. In the following only two-dimensional, incompressible, inviscid flows are considered. There are no assumptions made on the geometry chosen. The basic philosophy being that one first splits the body geometrically

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into a set of panels and on the surface of each of these panels one distributes some kind of singularity distribution. Normally the body is reduced to a set of piecewise linear elements. Typically used singularity distributions are constant, linear and quadratic distributions of sources, vorticity and doublets. For a constant distribution of singularity, given a set of N panels this results in N unknowns. In order to solve for the singularity distribution one must specify N conditions to make the problem determinable. Once this is done, one solves a matrix to determine the unknown distribution of singularity. The conditions are applied at certain control points and can be specified in two ways, a velocity boundary condition (this is called the Neumann condition) and by the specification of the potential inside the body (the Dirichlet condition). The obvious disadvantage with the Dirichlet method is that one cannot solve for flow past thickness less bodies. However for closed bodies they produce very good results at low panel densities.

Yon [7] performs an extensive study of nine different panel methods and reports finally that the combined constant source and doublet method with the Dirichlet formulation is the most robust from the practical requirements of speed and least sensitivity to panel densities. But for difficult geometries such as airfoils with cusped trailing edges or very thin airfoils, only the linear vortex Neumann formulation produced satisfactory results. This method was also found to be the only stable method that converged to the correct circulation around the lifting airfoils. Rajan [6] also shows that a vorticity distribution on the surface of a body is capable of explaining the kinematic motion of the rigid body in addition to solving the fluid flow. A distribution of doublets can also be used to solve for lifting bodies, however, it can be easily shown that a polynomial distribution of doublets of order n can be reduced to a distribution of vorticity of order $n-1$. Due to the above advantages of the linear vortex method and since the current work is interested in an accurate methodology for generalised bodies, both closed and open, thin and thick, and both lifting and non-lifting, a linear distribution of vorticity on the panels is chosen and the Neumann boundary condition (no penetration condition) is satisfied at the center of each panel.

Though panel methods provide very accurate results at and around the control points where the boundary condition is applied, they usually have problems near the edges where two piecewise linear panels intersect. Normally one is not concerned with these edge effects since one is interested in the distribution of pressure/velocity only on the control points so as to compute the load on the body. A new panel method that does not have this edge effect is developed in this work.

First, the velocity and potential field introduced by a linear panel with a linear distribution of vorticity is derived and studied. The fact that the linearisation of the geometry is the source of the error near the edge is demonstrated and subsequently solved by using cubic panels, the governing equations for which are derived in this work.

2. The Flat Panel Technique

The figure 1 shows a flat panel. The panel is first rotated about its first point, z_1, by an angle $-\theta$ and then shifted by a distance $-z_1$ to orient the panel along the x axis with its start point at the origin. The z' plane is the local coordinate system with respect to the panel as shown in Fig. 1. The point P moves to the corresponding point P' in the local coordinate plane.

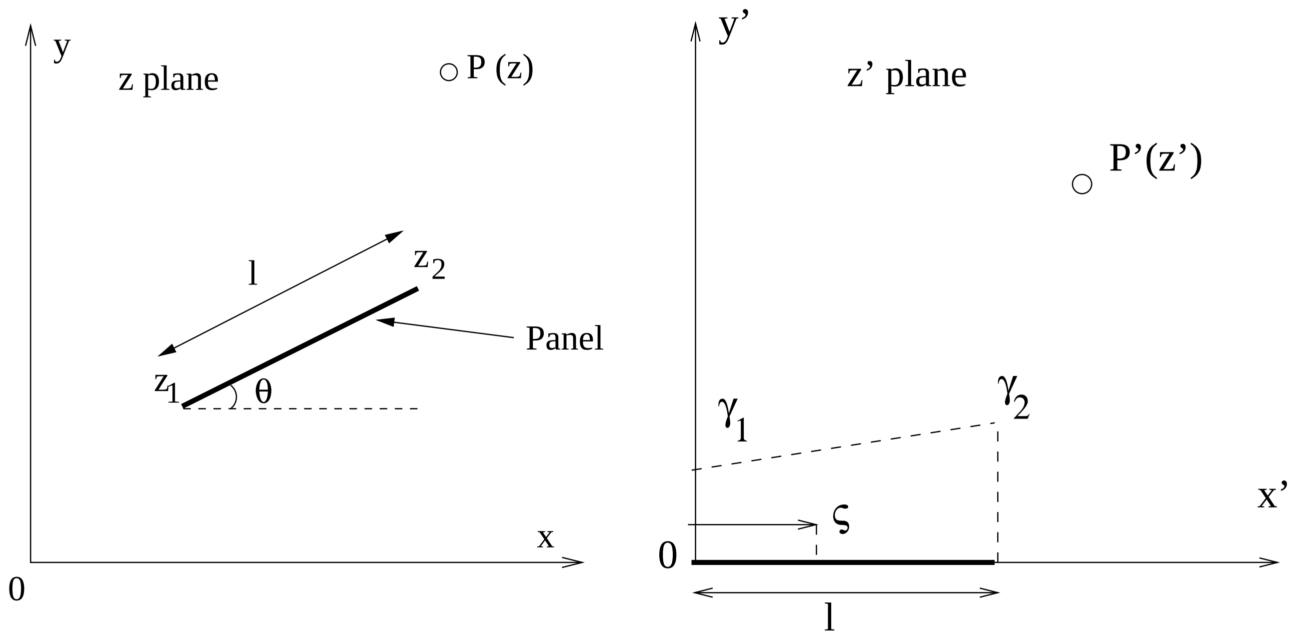

Figure 1: Sketch of a single flat panel in the z plane and its counterpart in the z' plane.

The vorticity γ at any point on the panel (ζ) is expressed as

$$\gamma = \gamma_1 + \frac{\gamma_2 - \gamma_1}{l}\zeta$$

Now in the local coordinate system the velocity due to the panel on the point P' can be easily obtained at any point z' as follows

$$V(z') = u' - iv' = \frac{-i}{2\pi}\int_0^l \frac{\gamma \, d\zeta}{z' - \zeta}$$
$$V(z') = \frac{-i}{2\pi}\int_0^l \frac{\gamma_1 + \frac{(\gamma_2-\gamma_1)\zeta}{l}}{z' - \zeta} d\zeta$$

where $i = \sqrt{-1}$ and l is the length of the panel. Integrating which we get

$$V(z') = u' - iv' = \frac{-i\gamma_1}{2\pi}\left[\left(\frac{z'}{l} - 1\right)\ln\left(\frac{z' - l}{z'}\right) + 1\right] - \frac{-i\gamma_2}{2\pi}\left[\frac{z'}{l}\ln\left(\frac{z' - l}{z'}\right) + 1\right] \quad (1)$$

Now rotating the velocity in the z' plane back to the z plane we obtain the velocity due to the panel at the z plane by noting that $\frac{dz'}{dz} = e^{-i\theta}$

$$V(z) = u - iv = \frac{-i}{2\pi}\left\{\gamma_1\left[\left(\frac{z'}{l} - 1\right)\ln\left(\frac{z' - l}{z'}\right) + 1\right] - \gamma_2\left[\frac{z'}{l}\ln\left(\frac{z' - l}{z'}\right) + 1\right]\right\} e^{-i\theta} \quad (2)$$

where $z' = (z - z_1)e^{-i\theta}$, z_1 is the first point of the panel and θ is its angle with respect to the x axis.

Given this velocity it is easy to solve for the flow past an arbitrary panelised body. If there are N panels for a closed body there are $2N$ values of γ to be computed. However, if it is required that the gamma distribution be continuous, then N additional conditions are obtained, therefore there are effectively only N unknowns. By choosing to satisfy the no-penetration condition at the center of each panel N more conditions are obtained. This is done by imposing the condition that $\overline{V_{total}} \cdot e^{i\theta_i} = 0$ where $\overline{V_{total}}$ is the conjugate of the total velocity at a panel control point and θ_i is the angle of that panel and the dot product being appropriately defined. For open bodies an additional condition is required and it is possible to either specify a condition on the total circulation or apply the Kutta condition.

Considering equation 1, it is found that at the origin of the z' plane the the first term diverges. Similarly the second term diverges as one approaches the point $z' = l$ implying that the velocity for the panel in the z plane diverges as one approaches either of its ends. If one were to distribute sources instead of panels this problem remains since the only difference between the two velocities is a multiplicative factor of i. For a distribution of doublets the situation is worse since a doublet would have a singularity of the form $\frac{1}{z}$ unlike the existing $\ln(z)$ singularity. It may be felt that the effect of this singularity will be nullified for a panelised body due to the effect of the next panel that shares an end point with this panel and since the sign of the γ_2 coefficient is opposite that of the γ_1 term. This is valid for a flat object, however, for a curved body, the angle of rotation involved for each panel is different since the discretisation of the body is in terms of flat panels. In order to illustrate this the flow past a thickness less flat plate at an angle of attack of 90^o is computed with a linear vortex distribution such that the total circulation over the flat plate is zero. The exact γ distribution for this flow is known from which the distribution of the magnitude of the velocity along the top surface of the body can be computed and compared to a computed solution employing 200 equally spaced panels with a linear distribution of vorticity. If the flow has a speed w and the plate centered about the origin, oriented along the x axis and having a half span of b $(b = l/2)$ then its γ distribution is given by $\gamma = 2 * w * x/\sqrt{b^2 - x^2}$ and due to symmetry the velocity distribution (which will be along the x axis) is given as $u = w * x/\sqrt{b^2 - x^2}$. For the comparison, $w = 1$ unit/sec and $l = 0.5$ units, are chosen. The figure 2 plots the corresponding curves for half the span. The velocity plot here due to the panels includes points that are just above the edges of the panel. Clearly no edge effects are seen and apart from the reasonably small error near the tips of the panel where the vorticity goes to infinity, the velocity magnitude elsewhere compares almost exactly showing that for flat objects the linear discretisation of the geometry does not produce any edge effects. It is to be noted that the error near the edges of the flat plate can be removed by using panels that are spaced according to a sinusoidal function.

If a curved body is considered, due to the difference in the angle of each panel and the singularity at its edge the velocity diverges as one approaches the edges. The natural solution for this problem is to discretise the body geometry in terms of cubic panels.

3. The Cubic Panel Technique

As was seen earlier if a curved body geometry is discretised in terms of piecewise linear panels, the velocity field diverges at the edges of the panels. Since this is due to the mismatch of the slopes of two adjacent panels, it is required that the the common end point as well as the slopes of two adjacent panels be matched. This requires four conditions to be applied to each panel and

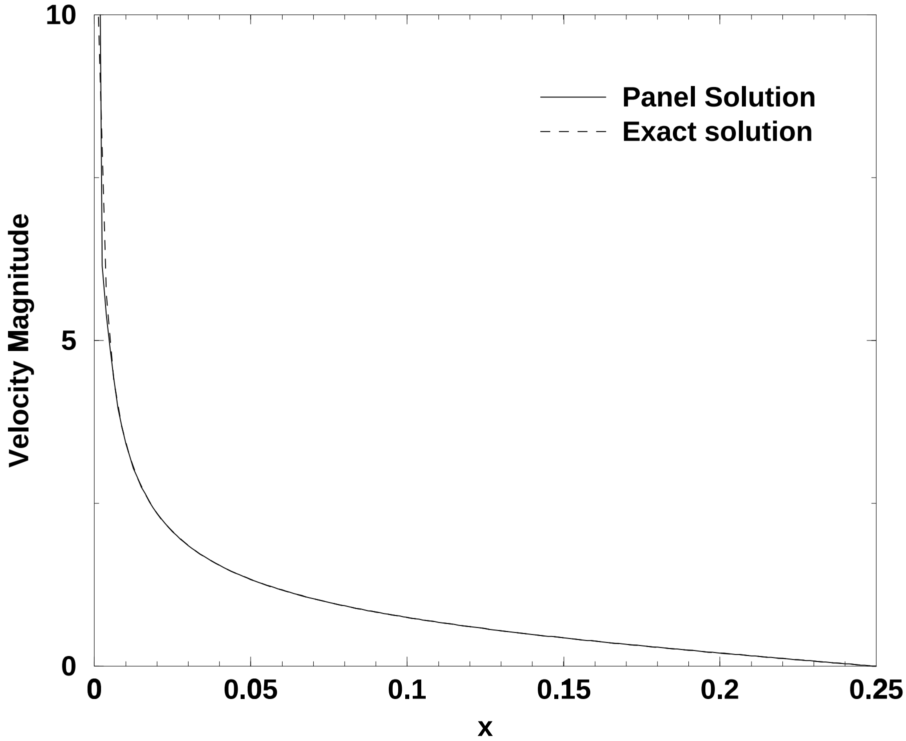

Figure 2: Exact and computed magnitude of velocity along the surface of a flat plate at an angle of attack $90°$.

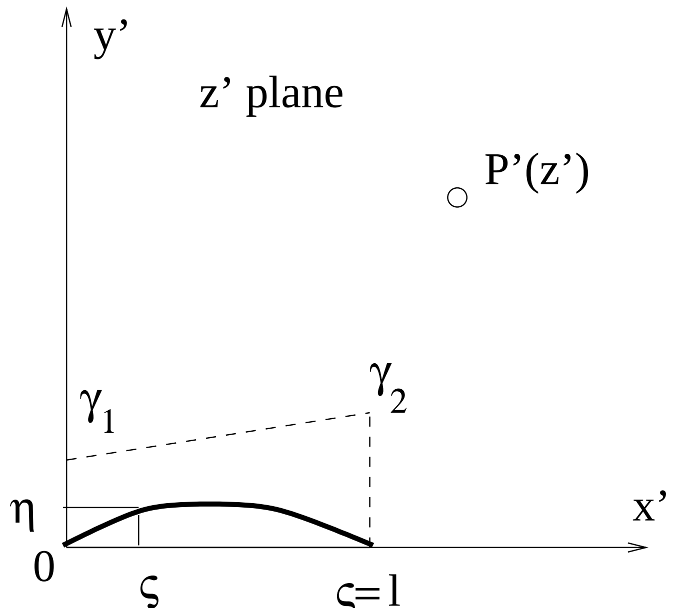

Figure 3: Sketch of a single cubic panel having a chord length l in the z' plane.

hence a piecewise cubic panel method is the right choice. Figure 3 shows a schematic of a cubic panel in the panel coordinates (z' plane). η is the height of the panel at the x location ζ. The panel is oriented such that its end points are along the x axis. Since the geometry is cubic, η in the panel coordinates is given by

$$\eta = a_1\zeta + a_2\zeta^2 + a_3\zeta^3 \tag{3}$$

where a_1, a_2 and a_3 are coefficients that depend on the both the slopes of the panel and the chord length, l. Depending on the implementation the values for these coefficients can be easily found. In order to simplify the equations the γ distribution is made linear with respect to ζ and not the arc length of the cubic panel. However the location of the distribution is along the cubic panel and not on its chord. The velocity due to the panel in the panel coordinates is given by

$$V(z') = u' - iv' = \frac{-i}{2\pi}\int_0^l \frac{(\gamma_1 + k\zeta)}{z' - (\zeta + i\eta)} d\zeta \tag{4}$$

by substituting equation 3 and simplifying the resulting expression the following is obtained

$$V(z') = \frac{k}{2\pi a_3}\int_0^l \frac{(\zeta + \gamma_1/k)}{\left(\zeta^3 + \frac{a_2}{a_3}\zeta^2 + \frac{(a_1 - i)}{a_3}\zeta + \frac{iz'}{a_3}\right)} d\zeta \tag{5}$$

where $k = (\gamma_2 - \gamma_1)/l$. The integral in equation 5 is not integrable directly. In order to solve it the cubic in the denominator is reduced as follows

$$\zeta^3 + \frac{a_2}{a_3}\zeta^2 + \frac{(a_1 - i)}{a_3} + \frac{iz'}{a_3}\zeta = (\zeta - a)(\zeta - b)(\zeta - c)$$

where a, b and c are the complex cube roots of the cubic. These roots can be computed numerically. The numerical procedure is standard and given in [5]. Given the roots one can find the velocity field due to the panels by performing the integration using the method of partial fractions. After integration and simplification the velocity due to the cubic panel is obtained as

$$\begin{aligned} V(z') \;=\; & \frac{-\gamma_2}{2\pi a_3 l}\left[\frac{a\log\left(\frac{a-l}{a}\right)}{(a-c)(a-b)} + \frac{b\log\left(\frac{b-l}{b}\right)}{(b-c)(b-a)} + \frac{c\log\left(\frac{c-l}{c}\right)}{(c-a)(c-b)}\right] \\ & - \frac{\gamma_1}{2\pi a_3 l}\left[\frac{(l-a)\log\left(\frac{a-l}{a}\right)}{(a-c)(a-b)} + \frac{(l-b)\log\left(\frac{b-l}{b}\right)}{(b-c)(b-a)} + \frac{(l-c)\log\left(\frac{c-l}{c}\right)}{(c-a)(c-b)}\right] \end{aligned} \tag{6}$$

where a, b and c are the complex cube roots of the cubic denominator in equation 5 that are computed numerically and a_1, a_2 and a_3 are the cubic coefficients of the panel that depend on both the slopes of the panel and the chord length, l. It must be noted that the cube roots involve z' implicitly. Using eqn. 6 the velocity due to a panel with an angle θ can be found just as in the case of the linear panel and is as follows

$$V(z) = V(z')e^{-i\theta} \tag{7}$$

Hence using equations 6 and 7 the velocity due to an arbitrary panel can be found. Using this newly developed velocity field the flow past any body can be computed. It must be noted that for each and every point z one needs to solve the cubic equation and obtain the cube roots. In a similar manner it is possible to derive the complex potential Φ due to a cubic panel. In the z' plane Φ can be obtained as follows

$$\Phi = \phi + i\psi = \frac{-ik}{2\pi}\int_0^l (\zeta + \frac{\gamma_1}{k})\, ln\, (ia_3(\zeta - a)(\zeta - b)(\zeta - c))\, d\zeta$$

which upon simplification and integration becomes

$$\Phi = \frac{-il}{4\pi}(\gamma_1 + \gamma_2)\ln(-ia_3) + I_1 + I_2 + I_3 \tag{8}$$

where

$$\begin{aligned} I_1 &= \frac{-ik}{4\pi}\left[(l-a)^2(\ln(l-a) - 0.5) - a^2(\ln(-a) - 0.5)\right] \\ &\quad + \frac{-ik}{2\pi}\left[(a+d)(l-a)(\ln(l-a) - 1) + (a+d)a(\ln(-a) - 1)\right] \end{aligned}$$

and I_2 and I_3 are the same as I_1 with the a replaced with b and c respectively and

$$d = \frac{\gamma_1}{k}, \;\; k = (\gamma_2 - \gamma_1)/l$$

and $a,\ b,\ c$ are the same cube roots used in the velocity evaluation. Thus the complex potential at any point due to the cubic panel can be found.

4. Validation

In order to show that the cubic panel distribution does not produce any edge effects the case of a flow past a circular cylinder of unit radius at zero angle of attack is considered. The figure 4 plots the velocity magnitude computed by both the linear panel technique and the newly developed cubic method. The free stream velocity is chosen as 1 unit/sec and 200 equal sized panels are used in both the cases. The edge effect produced by the flat panel technique is clearly seen. It is also clear that using the cubic panels completely removes the problem.

5. Conclusions

The reason why edge effects are seen in panel methods is studied and a solution has also been presented. The solution is to discretise the body in terms of cubic panels and not flat panels. The equations for such a panel distribution have been successfully developed. It has also been shown that such a technique effectively removes the edge effect and therby significantly reduces error. Using this method, it becomes possible to produce a globally accurate velocity field for the flow past an arbitrary shaped two-dimensional body using a panel method.

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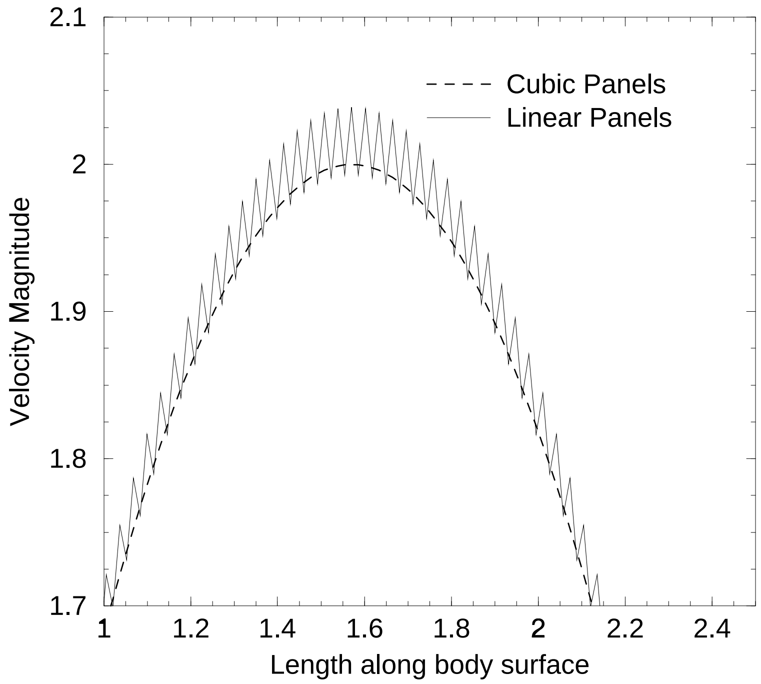

Figure 4: Illustration of the accuracy of the cubic panel method for the uniform flow past a 200 paneled cylinder

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